**Math 275 Elementary Differential Equations**

**Chapter 9 Variation of Parameters**

**9.2 Reduction of Order**

Consider the general second-order linear equation

$$y'' + py' + qy = R\,. \qquad (1)$$

Suppose we know a solution $y = y_1$ of the corresponding homogeneous equation

$$y'' + py' + qy = 0\,. \qquad (2)$$

Then an introduction of a new dependent variable $v$ by the substitution

$$y = y_1 v \qquad (3)$$

will lead to a solution of (1) as explained below.

From (3) it follows that

$$\begin{aligned} y' &= y_1 v' + y_1' v, \\ y'' &= y_1 v'' + 2y_1' v' + y_1'' v, \end{aligned}$$

so substitution of (3) into (1) yields

$$y_1 v'' + 2y_1' v' + y_1'' v + py_1 v' + py_1' v + qy_1 v = R,$$

or

$$y_1 v'' + (2y_1' + py_1) v' + (y_1'' + py_1' + qy_1) v = R. \qquad (4)$$

But $y = y_1$ is a solution of (2) i.e.

$$y_1'' + py_1' + qy_1 = 0$$

and (4) reduces to

$$y_1 v'' + (2y_1' + py_1) v' = R\,. \qquad (5)$$

Now let $v' = w$, so (5) becomes

$$y_1 w' + (2y_1' + py_1) w = R, \qquad (6)$$

a linear equation of first order in $w$.

By the usual method (integrating factor) we can find $w$ from (6). Then we can get $v$ by integration, and finally $y$.

**Examples**

**2)** $\left(D^2-5D+6\right)y=2e^x$

Solution:

$$\left(D^2-5D+6\right)y=2e^x$$
$$(D-2)(D-3)y=2e^x$$
$$m=2,3\Rightarrow y_c=c_1e^{2x}+c_2e^{3x}$$
$$\text{Let } y=e^{3x}\cdot v$$
$$y'=3e^{3x}v+e^{3x}v'$$
$$y''=9e^{3x}v+6e^{3x}v'+e^{3x}v''$$
$$9e^{3x}v+6e^{3x}v'+e^{3x}v''-15e^{3x}v-5e^{3x}v'+6e^{3x}v=2e^x$$
$$e^{3x}v''+e^{3x}v'=2e^x$$
$$v''+v'=2e^{-2x}\Rightarrow \text{I.F. } =e^{\int dx}=e^x$$
$$e^xv''+e^xv'=2e^{-x}$$
$$v'e^x=-2e^{-x}+c_1$$
$$v'=-2e^{-2x}+c_1e^{-x}$$
$$v=e^{-2x}+c_1e^{-x}+c_2$$
$$y=e^{3x}v=e^x+c_1e^{2x}+c_2e^{3x}$$

**Note**: #2 could have been solved by the method of undetermined coefficients.

**6)** $\left(D^2+1\right)y=\sec^3 x$

Solution:

$$\left(D^2+1\right)y=\sec^3 x$$
$$m=\pm i\Rightarrow y_c=c_1\cos x+c_2\sin x$$
$$\text{Let } y=v\sin x$$
$$y'=v\cos x+v'\sin x$$
$$y''=-v\sin x+2v'\cos x+v''\sin x$$
$$-v\sin x+2v'\cos x+v''\sin x+v\sin x=\sec^3 x$$
$$v''\sin x+2v'\cos x=\sec^3 x$$
$$v''+2v'\cot x=\frac{\sec^3 x}{\sin x}$$
$$\text{IF}=e^{2\int \cot x dx}=e^{2\ln|\sin x|}=\sin^2 x$$
$$v''\sin^2 x+2v'\cos x\sin x=\sec^3 x\sin x$$
$$v'\sin^2 x=\int \tan x\sec^2 x dx$$
$$v'\sin^2 x=\tfrac{1}{2}\tan^2 x+c_1$$
$$v'=\tfrac{1}{2}\sec^2 x+c_1\csc^2 x$$
$$v=\tfrac{1}{2}\tan x+c_1\cot x+c_2$$
$$y=v\sin x=\tfrac{1}{2}\sin x\tan x+c_1\cos x+c_2\sin x$$

**12)** Verify that $y = e^x$ is a solution of the equation $(x-1)y'' - xy' + y = 0$. Use this fact to find the general solution of $(x-1)y'' - xy' + y = 1$.

Solution:

$(x-1)e^x - xe^x + e^x = 0 \Rightarrow e^x$ is a solution of the DE

Let $y = ve^x$.

$$y' = ve^x + v'e^x$$
$$y'' = ve^x + 2v'e^x + v''e^x$$

Substitute in the 2nd DE:

$$(x-1)y'' - xy' + y = 1$$
$$(x-1)\left(ve^x + 2v'e^x + v''e^x\right) - x\left(ve^x + v'e^x\right) + ve^x = 1$$
$$\cancel{xve^x} + 2xv'e^x + xv''e^x \cancel{-ve^x} - 2v'e^x - v''e^x \cancel{-xve^x} - xv'e^x + \cancel{ve^x} = 1$$
$$xv'e^x + xv''e^x - 2v'e^x - v''e^x = 1$$
$$(x-1)e^x v'' + (x-2)e^x v' = 1$$
$$v'' + \frac{x-2}{x-1}v' = \frac{1}{(x-1)e^x}$$
$$\text{I.F.} = e^{\int \frac{x-2}{x-1}dx} = e^{\int \frac{x-1-1}{x-1}dx} = e^{\int \left(1-\frac{1}{x-1}\right)dx} = e^{x-\ln|x-1|} = \frac{e^x}{x-1}$$
$$\frac{e^x}{x-1}v'' + \frac{e^x}{x-1}\frac{x-2}{x-1}v' = \frac{e^x}{x-1}\frac{1}{(x-1)e^x}$$
$$\frac{e^x}{x-1}v' = -\frac{1}{x-1} + c_1$$
$$v' = -e^{-x} + c_1 e^{-x}(x-1)$$
$$\begin{array}{ll} x-1 & e^{-x} \\ 1 & -e^{-x} \\ & e^{-x} \end{array}$$
$$v = e^{-x} - c_1 e^{-x}(x-1) - c_1 e^{-x} + c_2$$
$$y = ve^x = 1 - c_1(x-1) - c_1 + c_2 e^x$$
$$y = 1 + c_1 x + c_2 e^x$$

**9.3 Variation of Parameters**

In Section 9.2 we saw that if $y_1$ is a solution of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0\,, \qquad (1)$$

we can use it to determine the general solution of the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = R(x)\,. \qquad (2)$$

In the method of reduction of order we used the fact that if $y_1$ is a solution of (1), then the function $c_1 y_1$ is also a solution for an arbitrary constant $c_1$. We replaced the constant $c_1$ by a function

$v(x)$ and considered the possibility of the existence of a solution of (2) of the form $vy_1$ and this led us to a first-order linear equation in the variable $v'$ that we were able to solve.

Suppose now that we know the general solution of (1), i.e. suppose that

$$y_c = c_1 y_1 + c_2 y_2 \tag{3}$$

is a solution of (1), where $y_1$ and $y_2$ are linearly independent on an interval $a < x < b$. Let us see what happens if we replace both of the constants in (3) with functions of *x*. Consider

$$y = Ay_1 + By_2 \tag{4}$$

and try to determine $A(x)$ and $B(x)$ so that $Ay_1 + By_2$ is a solution of (2). It follows from (4) that

$$y' = Ay_1' + By_2' + A'y_1 + B'y_2 . \tag{5}$$

Choose

$$A'y_1 + B'y_2 = 0 . \tag{6}$$

Then from (5)

$$y'' = Ay_1'' + By_2'' + A'y_1' + B'y_2' . \tag{7}$$

Substitute (4), (5), and (7) into (2):

$$A(y_1'' + py_1' + qy_1) + B(y_2'' + py_2' + qy_2) + A'y_1' + B'y_2' = R(x) .$$

But $y_1$ and $y_2$ are solutions of the homogeneous equation (1), so that

$$A'y_1' + B'y_2' = R(x) . \tag{8}$$

Solve (6) and (8) for $A'$ and $B'$. The solution exists provided

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 .$$

But this determinant is precisely the Wronskian of $y_1$ and $y_2$ and since we assumed that these two are linearly independent on $a < x < b$, then the Wronskian is never 0 on that interval and we can find $A'$ and $B'$.

**Examples**

**2)** $\left(D^2+1\right)y=\csc x\cot x$

Solution:

$$\left(D^2+1\right)y=\csc x\cot x$$

$$m=\pm i\Rightarrow y_c=c_1\cos x+c_2\sin x$$

Let $y=A\cos x+B\sin x$

$$y'=-A\sin x+B\cos x+A'\cos x+B'\sin x$$

Set $A'\cos x+B'\sin x=0.$ (1)

$$y'=-A\sin x+B\cos x$$

$$y''=-A\cos x-B\sin x-A'\sin x+B'\cos x$$

Substitute $y$ and $y''$ in the DE:

$$-A\cos x-B\sin x-A'\sin x+B'\cos x+A\cos x+B\sin x=\csc x\cot x$$

$$-A'\sin x+B'\cos x=\csc x\cot x \qquad (2)$$

$(1)\sin x+(2)\cos x:$

$$A'\cos x\sin x+B'\sin^2 x=0$$

$$\underline{-A'\sin x\cos x+B'\cos^2 x=\csc x\cot x\cos x}$$

$$B'\left(\sin^2 x+\cos^2 x\right)=\cot^2 x$$

$$B'=\cot^2 x=\csc^2 x-1$$

$$B=-\cot x-x$$

$(1)\cos x-(2)\sin x:$

$$A'\cos^2 x+B'\sin x\cos x=0$$

$$\underline{A'\sin^2 x-B'\cos x\sin x=-\sin x\csc x\cot x}$$

$$A'=-\cot x$$

$$A=-\ln\left|\sin x\right|$$

$$y_p=-\cos x\ln\left|\sin x\right|+\sin x\left(-\cot x-x\right)$$

$$y=c_1\cos x+c_2\sin x-x\sin x-\cos x\ln\left|\sin x\right|$$

**12)** $\left(D^2-3D+2\right)y=\dfrac{e^{2x}}{1+e^{2x}}$

Solution:

$$\left(D^2-3D+2\right)y=\frac{e^{2x}}{1+e^{2x}}$$
$$m=1,2\Rightarrow y_c=c_1e^x+c_2e^{2x}$$
$$y=Ae^x+Be^{2x}$$
$$y'=Ae^x+2Be^{2x}+A'e^x+B'e^{2x}\rightarrow A'e^x+B'e^{2x}=0 \quad (1)$$
$$y''=Ae^x+4Be^{2x}+A'e^x+2B'e^{2x}$$
$$Ae^x+4Be^{2x}+A'e^x+2B'e^{2x}-3Ae^x-6Be^{2x}+2Ae^x+2Be^{2x}=\frac{e^{2x}}{1+e^{2x}}$$
$$A'e^x+2B'e^{2x}=\frac{e^{2x}}{1+e^{2x}} \quad (2)$$
$$(1)-(2):B'e^{2x}=\frac{e^{2x}}{1+e^{2x}}\Rightarrow B'=\frac{1}{1+e^{2x}}=\frac{e^{-2x}}{e^{-2x}+1}\Rightarrow B=-\frac{1}{2}\ln\left(e^{-2x}+1\right)$$
$$2(1)-(2):A'e^x=-\frac{e^{2x}}{1+e^{2x}}\Rightarrow A'=-\frac{e^x}{1+e^{2x}}\Rightarrow A=-\arctan e^x$$
$$y_p=-e^x\arctan e^x-\frac{1}{2}e^{2x}\ln\left(e^{-2x}+1\right)$$
$$y=y_c-e^x\arctan e^x-\frac{1}{2}e^{2x}\ln\left(e^{-2x}+1\right)$$

**9.4 Solution of $y''+y=f(x)$ no need really!**

Consider next the equation

$$\left(D^2+1\right)y=f(x), \quad (1)$$

in which all that we require of $f(x)$ is that it be integrable in the interval on which we seek a solution. We will apply the method of variation of parameters to solve (1). Put

$$y=A\cos x+B\sin x. \quad (2)$$

Then

$$y'=-A\sin x+B\cos x+A'\cos x+B'\sin x,$$

and if we choose

$$A'\cos x+B'\sin x=0 \quad (3)$$

we get

$$y''=-A\cos x-B\sin x-A'\sin x+B'\cos x. \quad (4)$$

From (1), (2), and (4) it follows that

$$-A'\sin x+B'\cos x=f(x). \quad (5)$$

(3) and (5) may be solved for $A'$ and $B'$:

$$A' = -f(x)\sin x \quad \text{and} \quad B' = f(x)\cos x .$$

Now

$$A = -\int_a^x f(\beta)\sin\beta d\beta \qquad (6)$$

$$B = \int_a^x f(\beta)\cos\beta d\beta \qquad (7)$$

for any $x$ in $a \le x \le b$.

The $A$ and $B$ of (6) and (7) may be inserted in (2) to give us the particular solution

$$\begin{aligned} y_p &= -\cos x\int_a^x f(\beta)\sin\beta d\beta + \sin x\int_a^x f(\beta)\cos\beta d\beta \\ &= \int_a^x f(\beta)(\sin x\cos\beta - \cos x\sin\beta)d\beta . \qquad (8) \end{aligned}$$

Hence,

$$y_p = \int_a^x f(\beta)\sin(x-\beta)\,d\beta , \qquad (9)$$

and we can write the general solution of (1) as

$$y = c_1\cos x + c_2\sin x + \int_a^x f(\beta)\sin(x-\beta)\,d\beta . \qquad (10)$$