Math 275 Elementary Differential Equations

Chapter 8 Nonhomogeneous Equations: Undetermined Coefficients

8.1 Construction of a Homogeneous Equation from a Specific Solution

In Section 6.6 we saw that the general solution of the equation

$$\left(b_0 D^n + b_1 D^{n-1} + \cdots + b_{n-1} D + b_n\right) y = R(x) \qquad (1)$$

is

$$y = y_c + y_p,$$

where y_c, the complementary function, is the general solution of the homogeneous equation

$$\left(b_0 D^n + b_1 D^{n-1} + \cdots + b_{n-1} D + b_n\right) y = 0 \qquad (2)$$

and y_p is any particular solution of the original equation (1).

Goal: To be proficient in writing a homogeneous DE of which a given function of proper form is a solution.

Recall that in solving homogeneous equations with constant coefficients, a term such as $c_1 e^{ax}$ occurred when the auxiliary equation $f(m) = 0$ had a root $m = a$, and then the operator $f(D)$ had a factor $(D - a)$. Likewise, $c_2 x e^{ax}$ appeared only when $f(D)$ contained the factor $(D-a)^2$, $c_3 x^2 e^{ax}$ appeared only when $f(D)$ contained the factor $(D-a)^3$, and so on. Such terms as $ce^{ax}\cos bx$ or $ce^{ax}\sin bx$ correspond to roots $m = a \pm ib$, or to a factor $\left[(D-a)^2 + b^2\right]$.

Exercises

Obtain in factored form a linear DE with real, constant coefficients that is satisfied by the given function.

2) $y = 7 - 2x + \frac{1}{2}e^{4x}$

Solution:

The roots of the auxiliary equation are 0, 0, and 4. Thus, the DE is

$$D^2 (D - 4) y = 0.$$

4) $y = x^2 - 5\sin 3x$

Solution:

The roots of the auxiliary equation are $0, 0, 0, \pm 3i$. Thus, the DE is

$$D^3\left(D^2 + 9\right) = 0.$$

6) $y = 3e^{2x}\sin 3x$

Solution:

The roots of the AE are $2 \pm 3i$. Thus, the DE is

$$\left(D-(2+3i)\right)\left(D-(2-3i)\right)y = 0$$
$$\left[(D-2)^2 - 9i^2\right]y = 0$$
$$\left(D^2 - 4D + 13\right)y = 0$$

10) $y = \sin 2x + 3\cos 2x$

Solution:

The roots of the AE are $\pm 2i$. Thus, the DE is

$$\left(D^2 + 4\right)y = 0 .$$

List the roots of the AE for a homogeneous linear equation with real, constant coefficients that has the given function as a particular solution.

16) $y = x^2 e^{-x} + 4e^x$

Solution:

$-1,-1,-1,1$

18) $y = 3e^{-x}\cos 4x + 15e^{-x}\sin 4x$

Solution:

$-1 \pm 4i$

30) $y = \cos^2 x$

Solution:

$y = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

roots are $0, \ \pm 2i$

8.2 Solution of a Nonhomogeneous Equation

Let us consider a simple numerical example.

Consider the equation

$$D^2(D-1)y = 3e^x + \sin x . \tag{1}$$

The complementary function may be determined at once from the roots

$$m = 0,0,1 \tag{2}$$

of the auxiliary equation, namely,

$$y_c = c_1 + c_2 x + c_3 e^x . \tag{3}$$

Since the general solution of (1) is

$$y = y_c + y_p \, ,$$

where y_c is as given in (3) and y_p is any particular solution of (1), it remains to find a particular solution of (1).

The right side of (1),

$$R(x) = 3e^x + \sin x \, , \tag{4}$$

is a particular solution of a homogeneous linear DE whose auxiliary equation has the roots

$$m' = 1, \ \ \pm i \, . \tag{5}$$

Therefore, the function R is a particular solution of the equation

$$(D-1)\left(D^2+1\right)R = 0 \, . \tag{6}$$

We wish to convert (1) to a homogeneous linear DE with constant coefficients because we know how to solve such an equation. But by (6), the operator $(D-1)\left(D^2+1\right)$ will annihilate the right member of (1). Therefore we apply the operator to both sides of (1) and get

$$(D-1)\left(D^2+1\right)D^2\left(D-1\right)y = 0 \, . \tag{7}$$

Any solution of (1) must be a particular solution of (7). The general solution of (7) can be written at once from the roots of its auxiliary equation and is given by

$$y = c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 \cos x + c_6 \sin x \, . \tag{8}$$

But the desired general solution of (1) is

$$y = y_c + y_p \, , \tag{9}$$

where

$$y_c = c_1 + c_2 x + c_3 e^x \, ,$$

the c_1, c_2, c_3 being arbitrary constants as in (8). Thus there must exist a particular solution of (1) containing at most the remaining terms in (8). Using different letters as coefficients to emphasize that they are not arbitrary, we conclude that (1) has a particular solution

$$y_p = Axe^x + B\cos x + C\sin x \, . \tag{10}$$

We now have only to determine the numerical coefficients *A*, *B*, *C* by direct use of the original equation

$$D^2\left(D-1\right)y = 3e^x + \sin x \, . \tag{1}$$

From (10) it follows that

$$Dy_p = A\left(xe^x + e^x\right) - B\sin x + C\cos x,$$
$$D^2 y_p = A\left(xe^x + 2e^x\right) - B\cos x - C\sin x,$$
$$D^3 y_p = A\left(xe^x + 3e^x\right) + B\sin x - C\cos x.$$

Substitution of y_p into (1) then yields

$$Ae^x + (B+C)\sin x + (B-C)\cos x = 3e^x + \sin x. \tag{11}$$

Because (11) is to be an identity and because $e^x, \sin x,$ and $\cos x$ are linearly independent, the corresponding coefficients in the two members of (11) must be equal, that is,

$$\begin{aligned} &A = 3 \\ &B + C = 1 \\ &B - C = 0 \end{aligned}$$

giving $A = 3,\ B = \frac{1}{2},\ C = \frac{1}{2}$. Returning to (10) we find that a particular solution of (1) is

$$y_p = 3xe^x + \frac{1}{2}\cos x + \frac{1}{2}\sin x.$$

The general solution of the original equation is obtained by adding y_c and y_p:

$$y = c_1 + c_2 x + c_3 e^x + 3xe^x + \frac{1}{2}\cos x + \frac{1}{2}\sin x. \tag{12}$$

Summary of the steps:

(a) Find the values of m and m'.
(b) Write y_c and y_p.
(c) Substitute y_p into (1), equate corresponding coefficients, and obtain the numerical values of the coefficients in y_p.
(d) Write the general solution of (1).

8.3 The Method of Undetermined Coefficients

Let $f(D)$ be a polynomial in the operator D. Consider the equation

$$f(D)y = R(x). \tag{1}$$

Let the roots of the auxiliary equation $f(m) = 0$ be

$$m = m_1, m_2, \ldots, m_n. \tag{2}$$

The general solution of (1) is

$$y = y_c + y_p\,, \qquad (3)$$

where y_c can be obtained at once from the values of m in (2) and where $y = y_p$ is any particular solution of (1).

Now suppose $R(x)$ itself is a particular solution of some homogeneous linear DE with constant coefficients,

$$g(D)R = 0\,, \qquad (4)$$

whose auxiliary equation has the roots

$$m' = m_1', m_2', \ldots, m_k'\,. \qquad (5)$$

The DE

$$g(D)f(D)y = 0 \qquad (6)$$

has as the roots of its auxiliary equation the values of m from (2) and m' from (5). Hence the general solution of (6) contains the y_c of (3) and so is of the form

$$y = y_c + y_q\,.$$

But any particular solution of (1) must satisfy (6). Now, if

$$f(D)(y_c + y_q) = R(x)\,,$$

then $f(D)y_q = R(x)$ because $f(D)y_c = 0$. Then deleting the y_c from the general solution of (6) leaves a function y_q that for some numerical values of its coefficients must satisfy (1); i.e. the coefficients in y_q can be determined so that $y_q = y_p$.

Exercises

2) $(D^2 - 6D + 9)y = e^x$

Solution:

We have $m = 3,3$ and $m' = 1$. Then we have

$$y_c = c_1e^{3x} + c_2xe^{3x},$$
$$y_p = Ae^x,$$

where c_1 and c_2 are arbitrary constants, and A is to be determined numerically so that y_p will satisfy the equation. Now,

$$Dy_p = Ae^x$$
$$D^2y_p = Ae^x$$

and substituting these in the DE we get

$$\left(D^2-6D+9\right)y=e^x$$
$$Ae^x-6Ae^x+9Ae^x=e^x$$
$$4A=1$$
$$A=1/4$$

and thus the general solution is

$$y=c_1e^{3x}+c_2xe^{3x}+\frac{1}{4}e^x.$$

4) $\left(D^2+3D+2\right)y=1+3x+x^2$

Solution:

$$\left(D^2+3D+2\right)y=1+3x+x^2$$
$$m=-1,-2 \qquad m'=0,0,0$$
$$y_c=c_1e^{-x}+c_2e^{-2x}$$
$$y_p=A+Bx+Cx^2$$
$$Dy_p=B+2Cx$$
$$D^2y_p=2C$$
$$2C+3B+6Cx+2A+2Bx+2Cx^2=1+3x+x^2$$
$$2A+3B+2C=1\Rightarrow A=0$$
$$2B+6C=3\Rightarrow B=0$$
$$2C=1\Rightarrow C=1/2$$
$$y=c_1e^{-x}+c_2e^{-2x}+\tfrac{1}{2}x^2$$

10) $\left(D^2-D-2\right)y=6x+6e^{-x}$

Solution:

$$\left(D^2-D-2\right)y=6x+6e^{-x}$$
$$(D-2)(D+1)y=6x+6e^{-x}$$
$$m=2,-1 \qquad m'=0,0,-1$$
$$y_c=c_1e^{2x}+c_2e^{-x}$$
$$y_p=A+Bx+Cxe^{-x}$$
$$Dy_p=B-Cxe^{-x}+Ce^{-x}$$
$$D^2y_p=Cxe^{-x}-2Ce^{-x}$$
$$Cxe^{-x}-2Ce^{-x}-B+Cxe^{-x}-Ce^{-x}-2A-2Bx-2Cxe^{-x}=6x+6e^{-x}$$
$$-3Ce^{-x}-2Bx-B-2A=6x+6e^{-x}$$
$$-B-2A=0\Rightarrow A=3/2$$
$$-2B=6\Rightarrow B=-3$$
$$-3C=6\Rightarrow C=-2$$
$$y=c_1e^{2x}+c_2e^{-x}+\tfrac{3}{2}-3x-2xe^{-x}$$

12) $y'' - 4y' + 3y = 2\cos x + 4\sin x$

Solution:

$$y'' - 4y' + 3y = 2\cos x + 4\sin x$$
$$\left(D^2 - 4D + 3\right)y = 2\cos x + 4\sin x$$
$$(D-1)(D-3)y = 2\cos x + 4\sin x$$
$$m = 1,3 \qquad m' = \pm i$$
$$y_c = c_1 e^x + c_2 e^{3x}$$
$$y_p = A\cos x + B\sin x$$
$$y_p' = -A\sin x + B\cos x$$
$$y_p'' = -A\cos x - B\sin x$$
$$-A\cos x - B\sin x + 4A\sin x - 4B\cos x + 3A\cos x + 3B\sin x = 2\cos x + 4\sin x$$
$$2A - 4B = 2 \longrightarrow A - 2B = 1$$
$$4A + 2B = 4 \longrightarrow \underline{4A + 2B = 4}$$
$$5A \qquad = 5 \Rightarrow A = 1 \Rightarrow B = 0$$
$$y = c_1 e^x + c_2 e^{3x} + \cos x$$

8.4 Solution by Inspection

It is sometimes easy to obtain a particular solution of

$$\left(b_0 D^n + b_1 D^{n-1} + \cdots + b_{n-1} D + b_n\right) y = R(x) \tag{1}$$

by inspection.

If $R(x) = R_0$, a constant, and if $b_n \neq 0$, then

$$y_p = \frac{R_0}{b_n} \tag{2}$$

is a particular solution of

$$\left(b_0 D^n + b_1 D^{n-1} + \cdots + b_{n-1} D + b_n\right) y = R_0 . \tag{3}$$

Now suppose $b_n = 0$ in (3). Let $D^k y$ be the lowest-ordered derivative that actually appears in (3). Then we may write (3) as

$$\left(b_0 D^n + b_1 D^{n-1} + \cdots + b_{n-k} D^k\right) y = R_0 . \tag{4}$$

Now $D^k x^k = k!$, a constant, so that all higher derivatives of x^k are zero. Then (4) has a solution

$$y_p = \frac{R_0 x^k}{k!\, b_{n-k}} . \tag{5}$$

Examples

Find a particular solution by inspection.

6) $\left(D^2+2D-3\right)y=6$

Solution:

$$\left(D^2+2D-3\right)y=6$$

$$y_p=\frac{6}{-3}=-2$$

10) $\left(D^3-9D\right)y=27$

Solution:

$$\left(D^3-9D\right)y=27$$

$$y_p=\frac{27x}{1!(-9)}=-3x$$

14) $\left(D^4+D^2\right)y=-12$

Solution:

$$\left(D^4+D^2\right)y=-12$$

$$y_p=\frac{-12x^2}{2!\cdot 1}=-6x^2$$

16) $\left(D^5-9D^3\right)y=27$

Solution:

$$\left(D^5-9D^3\right)y=27$$

$$y_p=\frac{27x^3}{3!(-9)}=-\frac{1}{2}x^3$$

26) $\left(D^2-1\right)y=\sin 2x$

Solution:

A particular solution must be proportional to $\sin 2x$.

$$y=A\sin 2x$$

$$Dy=2A\cos 2x$$

$$D^2y=-4A\sin 2x$$

$$(-4-1)A=1\Rightarrow A=-\tfrac{1}{5}$$

$$y=-\tfrac{1}{5}\sin 2x$$

18) $\left(D^2+4\right)y=10\cos 3x$

Solution:

A particular solution must be proportional to $\cos 3x$.

$$y = A\cos 3x$$
$$Dy = -3A\sin 3x$$
$$D^2 y = -9A\cos 3x$$
$$(-9+4)A = 10 \Rightarrow A = -2$$
$$y = -2\cos 3x$$

28) $\left(D^2+1\right)y = 5e^{-3x}$

Solution:

$$\left(D^2+1\right)y = 5e^{-3x}$$
$$y = Ae^{-3x}$$
$$y' = -3Ae^{-3x}$$
$$y'' = 9Ae^{-3x}$$
$$10A = 5 \Rightarrow A = 1/2$$
$$y = \tfrac{1}{2}e^{-3x}$$

30) $\left(D^2+1\right)y = 4e^{-2x}$

Solution:

$$\left(D^2+1\right)y = 4e^{-2x}$$
$$y = Ae^{-2x}$$
$$y' = -2Ae^{-2x}$$
$$y'' = 4Ae^{-2x}$$
$$(4+1)A = 4 \Rightarrow A = 4/5$$
$$y = \tfrac{4}{5}e^{-2x}$$

34) $\left(D^2+2D+1\right)y = 7e^{-2x}$

Solution:

$$\left(D^2+2D+1\right)y = 7e^{-2x}$$
$$y = Ae^{-2x}$$
$$y' = -2Ae^{-2x}$$
$$y'' = 4Ae^{-2x}$$
$$(4-4+1)A = 7 \Rightarrow A = 7$$
$$y = 7e^{-2x}$$

Note: If y_1 is a solution of $f(D)y = R_1(x)$ and y_2 is a solution of $f(D)y = R_2(x)$, then $y_p = y_1 + y_2$ is a solution of $f(D)y = R_1(x) + R_2(x)$. Thus, the task of finding a particular solution may be split into parts by treating separate terms of $R(x)$ independently. This is the basis for the "**method of superposition**".

22) $\left(D^2+2D+5\right)y=4e^x-10$

Solution:

$$\left(D^2+2D+5\right)y=4e^x$$

$$(1+2+5)A=4\Rightarrow A=1/2\Rightarrow y_1=\tfrac{1}{2}e^x$$

$$\left(D^2+2D+5\right)y=-10$$

$$y_2=\frac{-10}{5}=-2$$

$$y=\tfrac{1}{2}e^x-2$$