Math 275 Elementary Differential Equations

Chapter 7 Linear Equations with Constant Coefficients

7.2 The Auxiliary Equation: Distinct Roots

Any linear homogeneous differential equation with constant coefficients,

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 , \qquad (1)$$

may be written in the form

$$f(D)\, y = 0 , \qquad (2)$$

where $f(D)$ is a linear differential operator. As we saw in the preceding chapter, if m is any root of the algebraic equation $f(m) = 0$, then

$$f(D)\, e^{mx} = 0 ,$$

which means that $y = e^{mx}$ is a solution of (2). The equation

$$f(m) = 0 \qquad (3)$$

is called the **<u>auxiliary equation</u>** associated with (1) or (2).

The auxiliary equation for (1) is of degree n. Let its roots be $m_1, m_2, \ldots, m_n$. If these roots are all real and distinct, then the n solutions

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}, \quad \ldots, \quad y_n = e^{m_n x}$$

are linearly independent and the general solution of (1) can be written at once and is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x} ,$$

where $c_1, c_2, \ldots, c_m$ are arbitrary constants.

Examples

#6) $\left(D^3 - 3D^2 - 10D\right) y = 0$

Solution:

$$m^3 - 3m^2 - 10m = 0$$
$$m\left(m^2 - 3m - 10\right) = 0$$
$$m(m-5)(m+2) = 0$$
$$m = 0, 5, -2$$
$$y = c_1 + c_2 e^{5x} + c_3 e^{-2x}$$

#18) $\left(4D^4 - 16D^3 + 7D^2 + 4D - 2\right)y = 0$

Solution:

$$4m^4 - 16m^3 + 7m^2 + 4m - 2 = 0$$

4	−16	7	4	−2	1/2
	2	−7	0	2	
4	−14	0	4	0	−1/2
	−2	8	−4		
4	−16	8	0		

$$4m^2 - 16m + 8 = 0$$

$$m^2 - 4m + 2 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2(1)} = 2 \pm \sqrt{2}$$

$$y = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x} + c_3 e^{(2+\sqrt{2})x} + c_4 e^{(2-\sqrt{2})x}$$

7.3 The Auxiliary Equation: Repeated Roots

Suppose that in the equation

$$f(D)y = 0 \qquad (1)$$

the operator $f(D)$ has repeated factors, i.e. the auxiliary equation $f(m) = 0$ has repeated roots, say there are three equal roots $m_1 = m_2 = m_3 = b$. The corresponding part of the solution yielded by the method of Section 7.2 is

$$\begin{aligned} y &= c_1 e^{bx} + c_2 e^{bx} + c_3 e^{bx}, \\ y &= (c_1 + c_2 + c_3)e^{bx}. \end{aligned} \qquad (2)$$

Now (2) can be replaced by

$$y = c_4 e^{bx} \qquad (3)$$

with $c_4 = c_1 + c_2 + c_3$. Thus, we only get one solution (3) and this is because the three solutions are not linearly independent.

Goal: To develop a method for finding *n* linearly independent solutions corresponding to *n* equal roots of the auxiliary equation.

Suppose the auxiliary equation $f(m) = 0$ has the *n* roots

$$m_1 = m_2 = \ldots = m_n = b.$$

Then the operator $f(D)$ must have a factor $(D - b)^n$. To find *n* linearly independent *y*'s for which

$$(D - b)^n y = 0. \qquad (4)$$

Using result (6) near the end of Section 6.9 and writing $m = b$, we find that

$$(D-b)^n\left(x^k e^{bx}\right) = 0 \text{ for } k = 0,1,2,\ldots,(n-1). \qquad (5)$$

The functions $y = x^k e^{bx}$ where $k = 0,1,2,\ldots,(n-1)$ are linearly independent because, aside from the common factor e^{bx}, they contain only the respective powers $x^0, x^1, x^2, \ldots, x^{n-1}$ (Exercise 5, Section 6.9).

The general solution of (4) is

$$y = c_1 e^{bx} + c_2 x e^{bx} + \ldots + c_n x^{n-1} e^{bx}. \qquad (6)$$

Also, if $f(D)$ contains the factor $(D-b)^n$, then the equation

$$f(D)\, y = 0 \qquad (1)$$

can be written

$$g(D)(D-b)^n\, y = 0 \qquad (7)$$

where $g(D)$ contains all the factors of $f(D)$ except $(D-b)^n$. Then any solution of

$$(D-b)^n\, y = 0 \qquad (4)$$

is also a solution of (7) and thus of (1).

Now we can write the solution of (1) whenever the auxiliary equation has only real roots. Each root of the auxiliary equation is either distinct from all the other roots or is one of a set of equal roots. Corresponding to a root m_i distinct from all others, there is the solution

$$y_i = c_i e^{m_i x}, \qquad (8)$$

and corresponding to *n* equal roots $m_1, m_2, \ldots, m_n$, each equal to *b*, there are solutions

$$c_1 e^{bx}, c_2 x e^{bx}, \ldots, c_n x^{n-1} e^{bx}. \qquad (9)$$

The collection of solutions in (8) and (9) has the proper number of elements, a number equal to the order of the DE, because there is one root corresponding to each root of the auxiliary equation. The solutions thus obtained can be proved to be linearly independent.

Exercise

#6) $\left(D^3 - 3D^2 + 4\right) y = 0$

Solution:

$$m^3 - 3m^2 + 4 = 0$$

$$\begin{array}{rrrr|l} 1 & -3 & 0 & 4 & \underline{2} \\ & 2 & -2 & -4 & \\ \hline 1 & -1 & -2 & 0 & \end{array}$$

$$m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0 \Rightarrow m = 2, -1$$

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 x e^{2x}$$

7.4 A Definition of exp *z* for Imaginary *z*

The auxiliary equation may have imaginary roots so we need to define e^z for imaginary *z*.

Let $z = \alpha + i\beta$ with α and β real. We will require that

$$e^{\alpha + i\beta} = e^{\alpha} e^{i\beta}. \tag{1}$$

To e^{α} with α real, we attach the usual meaning.

Now consider $e^{i\beta}$, β real. In calculus, it is shown that for any real number *x*

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \ldots, \tag{2}$$

or

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

If we tentatively put $x = i\beta$ in (2) as a definition of $e^{i\beta}$, we get

$$e^{i\beta} = 1 + \frac{i\beta}{1!} + \frac{i^2\beta^2}{2!} + \cdots + \frac{i^n\beta^n}{n!} + \ldots. \tag{3}$$

Separating the even powers of β from the odd powers of β in (3) yields

$$\begin{aligned} e^{i\beta} = 1 &+ \frac{i^2\beta^2}{2!} + \frac{i^4\beta^4}{4!} + \cdots + \frac{i^{2k}\beta^{2k}}{(2k)!} + \ldots \\ &+ \frac{i\beta}{1!} + \frac{i^3\beta^3}{3!} + \cdots + \frac{i^{2k+1}\beta^{2k+1}}{(2k+1)!} + \ldots \end{aligned}$$

or

$$e^{i\beta} = \sum_{k=0}^{\infty} \frac{i^{2k}\beta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1}\beta^{2k+1}}{(2k+1)!}. \tag{4}$$

Now $i^{2k} = (-1)^k$, so we may write

$$e^{i\beta} = 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \cdots + \frac{(-1)^k \beta^{2k}}{(2k)!} + \ldots$$
$$+ i\left[\frac{\beta}{1!} - \frac{\beta^3}{3!} + \cdots + \frac{(-1)^k \beta^{2k+1}}{(2k+1)!} + \ldots\right],$$

or

$$e^{i\beta} = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k \beta^{2k+1}}{(2k+1)!}. \tag{5}$$

But the series on the right in (5) are precisely those for $\cos\beta$ and $\sin\beta$ as developed in calculus. Hence we are led to the result

$$e^{i\beta} = \cos\beta + i\sin\beta. \tag{6}$$

Note that at this stage the above manipulations have no meaning in themselves but have suggested a meaningful *definition* of $e^{\alpha+iB}$, namely,

$$e^{\alpha+i\beta} = e^{\alpha}\left(\cos\beta + i\sin\beta\right) \quad \text{when } \alpha \text{ and } \beta \text{ are real.} \tag{7}$$

Replacing β by $(-\beta)$ in (7) we get

$$e^{\alpha-i\beta} = e^{\alpha}\left(\cos\beta - i\sin\beta\right).$$

7.5 The Auxiliary Equation: Imaginary Roots

Consider a DE $f(D)y = 0$ for which the auxiliary equation $f(m) = 0$ has real coefficients. From elementary algebra we know that if the auxiliary equation has any imaginary roots, these roots must occur in conjugate pairs. Thus if

$$m_1 = a + ib$$

is a root of the equation $f(m) = 0$, with *a* and *b* real and $b \neq 0$, then

$$m_2 = a - ib$$

is also a root of $f(m) = 0$.

We can now construct in usable form solutions of

$$f(D)y = 0 \tag{1}$$

corresponding to imaginary roots of $f(m) = 0$. For since $f(m)$ is assumed to have real coefficients, any imaginary roots appear in conjugate pairs,

$$m_1 = a + ib \quad \text{and} \quad m_2 = a - ib.$$

Then, according to Section 7.4, (1) is satisfied by

$$y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x} . \qquad (2)$$

Taking x to be real along with a and b, we get from (2) the result

$$y = c_1 e^{ax}\left(\cos bx + i \sin bx\right) + c_2 e^{ax}\left(\cos bx - i \sin bx\right) . \qquad (3)$$

Now (3) may be written

$$y = \left(c_1 + c_2\right) e^{ax} \cos bx + i\left(c_1 - c_2\right) e^{ax} \sin bx.$$

Finally, let $c_1 + c_2 = c_3$ and $i\left(c_1 - c_2\right) = c_4$, where c_3 and c_4 are new arbitrary constants. Then (1) has the solutions

$$y = c_3 e^{ax} \cos bx + c_4 e^{ax} \sin bx,$$

corresponding to the two roots $m_1 = a + ib$ and $m_2 = a - ib \quad \left(b \neq 0\right)$ of the auxiliary equation.

Exercises

#4) $\left(D^2 + 9\right) y = 0$

Solution:

$m^2 + 9 = 0 \Rightarrow m = \pm 3i$

$y = c_1 \cos 3x + c_2 \sin 3x$

#10) $\left(D^4 - 2D^3 + 2D^2 - 2D + 1\right) y = 0$

Solution:

$m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$

1	−2	2	−2	1	|1
	1	−1	1	−1	
1	−1	1	−1	0	|1
	1	0	1		
1	0	1	0		

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$y = c_1 e^x + c_2 x e^x + c_3 \cos x + c_4 \sin x$

#12) $\left(2D^4 + 11D^3 - 4D^2 - 69D + 34\right) y = 0$

Solution:

$$2m^4 + 11m^3 - 4m^2 - 69m + 34 = 0$$

2	11	−4	−69	34	2
	4	30	52	−34	
2	15	26	−17	0	1/2
	1	8	17		
2	16	34	0		

$$2m^2 + 16m + 34 = 0$$

$$m^2 + 8m + 17 = 0 \Rightarrow m = -4 \pm i$$

$$y = c_1 e^{2x} + c_2 e^{\frac{1}{2}x} + c_3 e^{-4x}\cos x + c_4 e^{-4x}\sin x$$

7.6 A Note on Hyperbolic Functions

The hyperbolic sine function is defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}\,; \tag{1}$$

the hyperbolic cosine function is defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}\,. \tag{2}$$

It follows that

$$\sinh^2 x = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$$

and

$$\cosh^2 x = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right),$$

so

$$\cosh^2 x - \sinh^2 x = 1. \tag{3}$$

From the definition we see that

$$y = \sinh u$$

is equivalent to

$$y = \frac{1}{2}\left(e^{u} - e^{-u}\right).$$

Hence, if u is a function of x, then

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{u} + e^{-u}\right)\frac{du}{dx},$$

that is,

$$\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}. \qquad (4)$$

The same method yields

$$\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}. \qquad (5)$$

Properties of the hyperbolic functions:

(a) $\cosh x \geq 1$ for all real *x*.
(b) The only real value of *x* for which $\sinh x = 0$ is $x = 0$.
(c) $\cosh(-x) = \cosh x$; that is, $\cosh x$ is an even function of *x*.
(d) $\sinh(-x) = -\sinh x$; $\sinh x$ is an odd function of *x*.

The period of the hyperbolic functions is $2\pi i$.

Since $D^2 \cosh ax = a^2 \cosh ax$ and $D^2 \sinh ax = a^2 \sinh ax$, it follows that both $\cosh ax$ and $\sinh ax$ are solutions of

$$\left(D^2 - a^2\right) y = 0, \qquad a \neq 0. \qquad (6)$$

Furthermore, the Wronskian of these two functions

$$W(x) = \begin{vmatrix} \cosh ax & \sinh ax \\ a\sinh ax & a\cosh ax \end{vmatrix} = a,$$

is not zero, so that $\cosh ax$ and $\sinh ax$ are linearly independent solutions of (6). Hence the general solution of (6) may be written as

$$y = c_1 \cosh ax + c_2 \sinh ax$$

instead of using the form

$$y = c_3 e^{ax} + c_4 e^{-ax}.$$