Math 275 Elementary Differential Equations

Chapter 6 Linear Differential Equations

6.1 The General Linear Equation

The general linear equation of order *n* is an equation that can be written as

$$b_0(x)\frac{d^n y}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x). \qquad (1)$$

Order 2 linear equation:

$$A(x)y'' + B(x)y' + C(x)y = R(x)$$

Def. (1) If the value of the function $R(x)$ is zero for all *x*, then the equation is called a **homogeneous** linear differential equation.

(2) If the coefficient functions $b_0, \ldots, b_n$ and the function *R* are continuous on an interval *I* and $b_0(x)$ is never zero on *I*, then (1) is said to be **normal** on *I*.

Example 1 $x^2 y' + y = e^x$

First order, linear, nonhomogeneous, and normal on any interval that does not contain $x = 0$

Example 2 $y'' + 2y' + 3xy = 0$

2^{nd} order, linear, homogeneous, and normal on any interval

Def. Given functions $y_1, y_2, \ldots, y_k$.

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_k y_k,$$

where $c_1, c_2, \ldots, c_k$ are constants, is called a **linear combination** of $y_1, y_2, \ldots, y_k$.

Theorem 6.1 Any linear combination of solutions of a homogeneous linear differential equation is also a solution.

Proof:

To prove that if y_1 and y_2 are solutions of the homogeneous equation

$$b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_{n-1}(x)y' + b_n(x)y = 0 \qquad (2)$$

and if c_1 and c_2 are constants, then

$$y = c_1 y_1 + c_2 y_2$$

is a solution of (2).

Now, if y_1 and y_2 are solutions of (2) then

$$b_0(x)y_1^{(n)} + b_1(x)y_1^{(n-1)} + \cdots + b_{n-1}(x)y_1' + b_n(x)y_1 = 0 \qquad (3)$$

and

$$b_0(x)y_2^{(n)} + b_1(x)y_2^{(n-1)} + \cdots + b_{n-1}(x)y_2' + b_n(x)y_2 = 0 \qquad (4)$$

Multiply (3) by c_1 and (4) by c_2 and add the results to get

$$\begin{aligned} b_0(x)\left(c_1y_1^{(n)} + c_2y_2^{(n)}\right) + b_1(x)\left(c_1y_1^{(n-1)} + c_2y_2^{(n-1)}\right) + \cdots \\ + b_{n-1}(x)(c_1y_1' + c_2y_2') + b_n(x)(c_1y_1 + c_2y_2) = 0 \end{aligned} \qquad (5)$$

Since $c_1y_1' + c_2y_2' = (c_1y_1 + c_2y_2)'$, and so on, (5) says that $c_1y_1 + c_2y_2$ is a solution of (2).

Note:

1. For the case $c_2 = 0$, the theorem says that any constant times a solution is also a solution.
2. If $y_1, y_2, \ldots, y_k$ are solutions of (2) and $c_1, c_2, \ldots, c_k$ are constants, then $y = c_1y_1 + c_2y_2 + \ldots + c_ky_k$ is a solution of (2).

6.2 An Existence and Uniqueness Theorem

Theorem 6.2 Given an *n*th-order linear differential equation

$$b_0(x)\frac{d^n y}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x) \qquad (1)$$

that is normal on an interval *I*. Suppose that x_0 is any number on the interval *I* and $y_0, y_1, \ldots, y_{n-1}$ are *n* arbitrary real numbers. Then a unique function $y = y(x)$ exists such that *y* is a solution of the differential equation on the interval and *y* satisfies the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

Exercises

Determine all intervals on which the equation is normal.

2) $(x^2 - 1)y'' + 6y = e^x$

Solution:

Normal everywhere except where $x = \pm 1$, thus $x < -1, \quad -1 < x < 1, \quad x > 1$.

Determine the unique solution of the initial value problem.

6) $y'' + 4y = 0,\ y(0) = 2,\ y'(0) = 4$. Use the fact that $\sin 2x$ and $\cos 2x$ are solutions of the DE.

Solution:

Since $\sin 2x$ and $\cos 2x$ are solutions then for arbitrary constants c_1 and c_2,

$$y = c_1 \sin 2x + c_2 \cos 2x$$

is also a solution. Using the initial condition $y(0) = 2$ we get

$$2 = c_1 \sin 0 + c_2 \cos 0 \Rightarrow c_2 = 2.$$

Using the initial condition $y'(0) = 4$ we get

$$4 = 2c_1 \cos 0 - c_2 \sin 0 \Rightarrow c_1 = 2.$$

Thus, the unique solution of the DE is

$$y = 2\sin 2x + 2\cos 2x.$$

8) $x^2 y'' + xy' - 9y = 0,\ \ y(1) = -1,\ \ y'(1) = 15$

Solution:

x^3 and x^{-3} are solutions of the DE and the DE is normal on $x < 0$ or $x > 0$.
Since $x_0 = 1$, choose the interval *I* to be $x > 0$.

$y = c_1 x^3 + c_2 x^{-3}$ is also a solution.
Using the initial conditions,

$$\begin{aligned} c_1 + c_2 &= -1 \quad \longrightarrow c_1 + c_2 = -1 \\ 3c_1 - 3c_2 &= 15 \longrightarrow \underline{c_1 - c_2 = 5} \\ &\qquad\qquad 2c_1 \qquad = 4 \Rightarrow c_1 = 2,\ \ c_2 = -3 \end{aligned}$$

Thus, $y = 2x^3 - 3x^{-3}$.

6.3 Linear Independence

Given the functions $f_1, f_2, \ldots, f_n$, if constants $c_1, c_2, \ldots, c_n$, not all zero, exist such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \qquad (1)$$

for all *x* in some interval $a \le x \le b$, then the functions $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** on that interval; otherwise, the functions are said to be **linearly independent**, i.e. (1) implies

$$c_1 = c_2 = \ldots = c_n = 0.$$

Note: If the functions of a set are linearly dependent, at least one of them is a linear combination of the others; if they are linearly independent, none of them is a linear combination of the others.

To develop sufficient conditions for linear independence:

6.4 The Wronskian

Goal: To obtain sufficient conditions for linear independence of n functions on an interval $a \le x \le b$.

Assume that each of the functions $f_1, f_2, \ldots, f_n$ is differentiable at least $(n-1)$ times in the interval $a \le x \le b$. Then from

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \qquad (1)$$

it follows by successive differentiation that

$$\begin{aligned} c_1 f_1' + c_2 f_2' + \cdots + c_n f_n' &= 0 \\ c_1 f_1'' + c_2 f_2'' + \cdots + c_n f_n'' &= 0 \\ &\vdots \\ c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \cdots + c_n f_n^{(n-1)} &= 0. \end{aligned}$$

For any fixed value of x in the interval $a \le x \le b$, the nature of the solutions of these n linear equations in $c_1, c_2, \ldots, c_n$ will be determined by the value of the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ & \vdots & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}. \qquad (2)$$

If $W(x_0) \ne 0$ for some x_0 in the interval $a \le x \le b$, then $c_1 = c_2 = \ldots = c_n = 0$, and thus the functions $f_1, f_2, \ldots, f_n$ are linearly independent on $a \le x \le b$.

The function $W(x)$ is called the **<u>Wronskian</u>** of the n functions $f_1, f_2, \ldots, f_n$.

Theorem 6.3 If on an interval $a \le x \le b$, $b_0(x) \ne 0$, $b_0, b_1, \ldots, b_n$ are continuous, and $y_1, y_2, \ldots, y_n$ are solutions of the equation

$$b_0 y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-1} y' + b_n y = 0, \qquad (3)$$

then $y_1, y_2, \ldots, y_n$ are linearly independent if and only if the Wronskian of $y_1, y_2, \ldots, y_n$ is different from 0 at at least one point on the interval $a \le x \le b$.

Example (Ex 6.3 p. 104)

$\cos 2x$, $\sin 2x$, $\sin(2x+3)$ are linearly dependent on any interval since

$\sin(2x+3) - \cos 3 \sin 2x - \sin 3 \cos 2x = 0$ for all x.

$$W(x)=\begin{vmatrix}\cos 2x & \sin 2x & \sin(2x+3)\\ -2\sin 2x & 2\cos 2x & 2\cos(2x+3)\\ -4\cos 2x & -4\sin 2x & -4\sin(2x+3)\end{vmatrix}$$

$$=\cos 2x\left[-8\cos 2x\sin(2x+3)+8\sin 2x\cos(2x+3)\right]$$
$$-\sin 2x\left[8\sin 2x\sin(2x+3)+8\cos 2x\cos(2x+3)\right]$$
$$+\sin(2x+3)\left[8\sin^2 2x+8\cos^2 2x\right]=0$$

Also, each function is a solution of $y'''+4y'=0$. Thus, $\cos 2x,\ \sin 2x,\ \sin(2x+3)$ are linearly dependent.

Example #2 p. 104 Show e^x, e^{2x}, e^{3x} are linearly independent.

Solution:

$$W(x)=\begin{vmatrix}e^x & e^{2x} & e^{3x}\\ e^x & 2e^{2x} & 3e^{3x}\\ e^x & 4e^{2x} & 9e^{3x}\end{vmatrix}=2e^{6x}\neq 0$$

6.5 General Solution of a Homogeneous Equation

Theorem 6.4 Let $\{y_1, y_2, \ldots, y_n\}$ be a linearly independent set of solutions of the homogeneous linear equation

$$b_0(x)y^{(n)}+b_1(x)y^{(n-1)}+\cdots+b_{n-1}(x)y'+b_n(x)y=0\,, \qquad (1)$$

for x on the interval $a\le x\le b$. Suppose further that the equation is normal on $a\le x\le b$. If ϕ is any solution of (1), valid on $a\le x\le b$, there exist constants $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n$ such that

$$\phi=\bar{c}_1y_1+\bar{c}_2y_2+\cdots+\bar{c}_ny_n\,. \qquad (2)$$

Note:

1. It is because of T6.4 that we define the **<u>general solution</u>** of (1) to be

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n\,, \qquad (3)$$

   where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

2. Each particular solution of (1) is a special case of the general solution.

   Proof:

   Consider the equation

$$b_0(x)y''+b_1(x)y'+b_2(x)y=0\,. \qquad (4)$$

   Let y_1 and y_2 be linearly independent solutions of (4) on the interval $a\le x\le b$. By T 6.3, there exists a number x_0 in the interval such that

$$W = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0 . \quad (5)$$

It follows that the system of equations

$$c_1 y_1(x_0) + c_2 y_2(x_0) = \phi(x_0)$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = \phi'(x_0)$$

has a unique solution $c_1 = \bar{c}_1,\ c_2 = \bar{c}_2$. That is,

$$\bar{c}_1 y_1(x_0) + \bar{c}_2 y_2(x_0) = \phi(x_0),$$
$$\bar{c}_1 y_1'(x_0) + \bar{c}_2 y_2'(x_0) = \phi'(x_0).$$

Now consider the function

$$f = \bar{c}_1 y_1 + \bar{c}_2 y_2 . \quad (6)$$

Because f is a linear combination of two solutions of (4) on the interval $a \le x \le b$, it is also a solution on that interval. Moreover,

$$f(x_0) = \bar{c}_1 y_1(x_0) + \bar{c}_2 y_2(x_0),$$
$$f'(x_0) = \bar{c}_1 y_1'(x_0) + \bar{c}_2 y_2'(x_0),$$

so that $f(x_0) = \phi(x_0)$ and $f'(x_0) = \phi'(x_0)$. It follows from the uniqueness theorem of Section 13.2 that f and ϕ are the same solution. That is,

$$\phi = \bar{c}_1 y_1 + \bar{c}_2 y_2 .$$

6.6 General Solution of a Nonhomogeneous Equation

Let y_p be any particular solution (not necessarily involving any arbitrary constants) of the equation

$$b_0(x) y^{(n)} + b_1(x) y^{(n-1)} + \cdots + b_{n-1}(x) y' + b_n(x) y = R(x) \quad (1)$$

and let y_c be a solution of the corresponding homogeneous equation

$$b_0(x) y^{(n)} + b_1(x) y^{(n-1)} + \cdots + b_{n-1}(x) y' + b_n(x) y = 0 . \quad (2)$$

Then

$$y = y_c + y_p \quad (3)$$

is a solution of (1) since

$$b_0 y^{(n)} + \cdots + b_n y = \left(b_0 y_c^{(n)} + \cdots + b_n y_c\right) + \left(b_0 y_p^{(n)} + \cdots + b_n y_p\right)$$
$$= 0 + R(x) = R(x)$$

If $y_1, y_2, \ldots, y_n$ are linearly independent solutions of (2), then

$$y_c = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n, \qquad (4)$$

in which the *c*'s are arbitrary constants, is the general solution of (2). The right member of (4) is called the **complementary function** for (1).

The general solution of the nonhomogeneous equation (1) is the sum of the complementary function and any particular solution. To justify this usage of the term *general solution*, we must show that if *f* is any solution of (1), then $f = y_c + y_p$ for some particular choice of the $c_1, \ldots, c_n$. We note that since *f* and y_p are both solutions of (1), $f - y_p$ is a solution of (2). Hence, by T6.4 of Section 6.5,

$$f - y_p = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

for some particular choice of the $c_1, \ldots, c_n$.

Ex1 (Ex 6.5 p.108) Find the general solution of $y'' = 4$.

Solution:

$1, x$ are linearly independent on any interval and are solutions of $y'' = 0$.

Hence, $y_c = c_1 + c_2 x$.

$2x^2$ is a particular solution so $y_p = 2x^2$.

Thus, the general solution is $y = c_1 + c_2 x + 2x^2$.

Ex 2 (Ex 6.6 p. 108) Find the general solution of $y'' - y = 4$.

Solution:

$y_p = -4$

$y_c = c_1 e^x + c_2 e^{-x}$

$y = c_1 e^x + c_2 e^{-x} - 4$

6.7 Differential Operators

Let *D* denote differentiation wrt *x*, D^2 differentiation twice wrt *x*, and so on; i.e. for positive integral *k*,

$$D^k y = \frac{d^k y}{dx^k}.$$

The expression

$$A = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n \qquad (1)$$

is called a **differential operator of order *n***. It may be defined as that operator which, when applied to any function *y*, yields the result

$$Ay = a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y \ . \qquad (2)$$

The coefficients $a_0, a_1, \ldots, a_n$ in the operator *A* may be functions of *x*, but most operators in the book will have constant coefficients.

Two operators *A* and *B* are said to be **equal** if and only if the same result is produced when each acts upon the function *y*, i.e. $A = B \Leftrightarrow Ay = By$ for all functions *y* possessing the derivatives necessary for the operations involved.

The **product** of two operators *A* and *B* is defined to be as that operator which produces the same result as is obtained by using the operator *B* followed by the operator *A*. Thus $ABy = A(By)$.
The product of two differential operators always exists and is a differential operator. For operators with *constant coefficients*, but not usually those with variable coefficients, it is true that $AB = BA$.

Ex 1 $A = D+3, \quad B = 2D-1$

$$By = (2D-1)y = 2y' - y$$
$$\begin{aligned} A(By) &= (D+3)(2y'-y) \\ &= 2Dy' + 6y' - Dy - 3y \\ &= 2y'' + 6y' - y' - 3y \\ &= 2y'' + 5y' - 3y \Rightarrow AB = 2D^2 + 5D - 3 \end{aligned}$$
$$\begin{aligned} B(Ay) &= (2D-1)(y'+3y) \\ &= 2Dy' + 6Dy - y' - 3y \\ &= 2y'' + 6y' - y' - 3y \\ &= 2y'' + 5y' - 3y \end{aligned}$$
$$BA = 2D^2 + 5D - 3 = AB$$

Ex 2 $A = xD+3, \quad B = D-1$

$$By = (D-1)y = y' - y$$
$$\begin{aligned} A(By) &= (xD+3)(y'-y) \\ &= xDy' + 3y' - xDy - 3y \\ &= xy'' + 3y' - xy' - 3y \\ &= xy'' + (3-x)y' - 3y \end{aligned}$$
$$AB = D^2 + (3-x)D - 3$$
$$Ay = (xD+3)y = xy' + 3y$$
$$\begin{aligned} B(Ay) &= (D-1)(xy'+3y) \\ &= D(xy') - xy' + 3Dy - 3y \\ &= y' + xy'' - xy' + 3y' - 3y \\ &= xy'' - (x-2)y' - 3y \end{aligned}$$
$$BA = xD^2 - (x-2)D - 3$$

6.8 The Fundamental Laws of Operation

Let A, B, and C be any differential operators as defined in Section 6.7. Differential operators satisfy the following:

(a) The commutative law of addition:

$$A+B=B+A$$

(b) The associative law of addition:

$$(A+B)+C=A+(B+C)$$

(c) The associative law of multiplication:

$$(AB)C=A(BC)$$

(d) The distributive law of multiplication wrt addition:

$$A(B+C)=AB+AC.$$

(e) If A and B are operators with *constant coefficients*, then

$$AB=BA.$$

Note:

1. Differential operators with constant coefficients satisfy all the laws of the algebra of polynomials wrt the operations of addition and multiplication.
2. If m and n are any two positive integers, then

$$D^m D^n = D^{m+n}$$

is a useful result that follows immediately from the definitions.

6.9 Some Properties of Differential Operators

Since for constant m and positive integer k,

$$D^k e^{mx} = m^k e^{mx}, \tag{1}$$

it is easy to find the effect that an operator has upon e^{mx}. Let $f(D)$ be a polynomial in D,

$$f(D)=a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n. \tag{2}$$

Then

$$f(D)e^{mx} = a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx},$$

so

$$f(D)e^{mx} = e^{mx} f(m). \tag{3}$$

If m is a root of the equation $f(m)=0$, then in view of (3),

$$f(D)e^{mx} = 0\,.$$

Next, consider the effect of the operator $D-a$ on the product of e^{ax} and a function of *y*. We have

$$\begin{aligned}(D-a)\left(e^{ax}y\right) &= D\left(e^{ax}y\right) - ae^{ax}y \\ &= e^{ax}Dy\end{aligned}$$

and

$$\begin{aligned}(D-a)^2\left(e^{ax}y\right) &= (D-a)\left(e^{ax}Dy\right) \\ &= e^{ax}D^2y\end{aligned}$$

Repeating the operation, we are led to

$$(D-a)^n\left(e^{ax}y\right) = e^{ax}D^ny\,.$$

Using the linearity of differential operators, we conclude that when $f(D)$ is a polynomial in *D* with constant coefficients, then

$$e^{ax}f(D)y = f(D-a)\left[e^{ax}y\right]. \qquad (5)$$

The relation (5) shows us how to shift an exponential factor from the left of a differential operator to the right of the operator.

Note (5) shows how to shift an exponential factor from the left of a DO to the right of a DO.

Ex 1 $f(D) = 3D^2 + 2D - 8$

$f(m) = 0$

$3m^2 + 2m - 8 = 0$

$(3m-4)(m+2) = 0$

$m_1 = 4/3, \quad m_2 = -2$

By (3), $\left(3D^2 + 2D - 8\right)e^{4/3x} = 0$ and $\left(3D^2 + 2D - 8\right)e^{-2x} = 0$.

That is, $y_1 = e^{4/3x}$ and $y_2 = e^{-2x}$ are solutions of

$3D^2 + 2D - 8 = 0$.

Ex 2 (Ex 6.10 p. 114)

Show that $(D-m)^n\left(x^k e^{mx}\right) = 0$ for $k = 0,1,\ldots,n-1$ (6)

Solution:

In (5) we let $f(D) = D^n$ and $y = x^k$. Then using the exponential shift we get

$$(D-m)^n\left(x^k e^{mx}\right) = e^{mx}D^n x^k\,.$$

But $D^n x^k = 0$ for $k = 0,1,2,\ldots,n-1$ and (6) follows.

Important Results for Chapter 7

$$f(D)e^{mx} = e^{mx}f(m) \tag{3}$$

$$e^{ax}f(D)y = f(D-a)\left[e^{ax}y\right] \tag{5}$$

$$(D-m)^n\left(x^k e^{mx}\right) = 0, \quad k = 0,1,\ldots,n-1 \tag{6}$$

Ex 3 exponential shift

Solve $(D+2)^3 y = 0$ (7)

Solution:

Multiply by e^{2x} :

$e^{2x}(D+2)^3 y = 0$

$D^3\left(e^{2x}y\right) = 0$ by (5)

Integrating 3 times

$e^{2x}y = c_1 + c_2 x + c_3 x^2$

$\Rightarrow y = \left(c_1 + c_2 x + c_3 x^2\right)e^{-2x}$ (8)

Note: Each of e^{-2x}, xe^{-2x}, x^2e^{-2x} is a solution of (7). It remains to show that the 3 functions are linearly independent. Then (8) gives the general solution of (7). (See Exercise 5 p. 115)