**Math 275 Elementary Differential Equations**

**Chapter 5 Additional Topics on Equations of Order One**

**5.1 Integrating Factors Found by Inspection**

Four exact differentials that occur frequently:

$$d(xy) = xdy + ydx$$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$$

$$d\left(\arctan\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$$

**Examples**

**1.** Solve $\left(y^2 - y\right)dx + xdy = 0$

Solution:

Regroup terms:

$$-(ydx - xdy) + y^2dx = 0$$

Divide by $y^2$ :

$$-\frac{ydx - xdy}{y^2} + 1dx = 0$$

$$-d\left(\frac{x}{y}\right) = -dx$$

$$\frac{x}{y} = x + c \quad \text{or} \quad y = \frac{x}{x+c}$$

**2.** (Exercise #4) $2tds + s\left(2 + s^2t\right)dt = 0$

Solution:

$$2tds + s\left(2 + s^2t\right)dt = 0$$

$$2(tds + sdt) + s^3tdt = 0$$

$$2d(st) + s^3tdt = 0$$

$$\frac{2}{s^3}d(st) + tdt = 0$$

We want to find an IF that will make the coefficient of $d(st)$ a function of *st*.

Assume the IF is $s^kt^n$ , *k* and *n* to be determined.

$$2s^{k-3}t^n d(st) + t^{n+1}dt = 0$$
$$k-3 = n \text{ and } k = 0 \Rightarrow n = -3$$
$$2(st)^{-3}\, d(st) + t^{-2}dt = 0$$
$$2\frac{(st)^{-2}}{-2} + \frac{t^{-1}}{-1} = c_1$$
$$-\frac{1}{(st)^2} - \frac{1}{t} = c_1$$
$$1 + s^2 t = cs^2 t^2$$

**5.2 The Determination of Integrating Factors**

Suppose that *u*, possibly a function of both *x* and *y*, is to be an integrating factor of

$$Mdx + Ndy = 0 \qquad (1)$$

Then the equation

$$uM\ dx + uN\ dy = 0 \qquad (2)$$

must be exact and

$$\frac{\delta}{\delta y}(uM) = \frac{\delta}{\delta x}(uN).$$

Thus, *u* must satisfy the partial differential equation

$$u\frac{\delta M}{\delta y} + M\frac{\delta u}{\delta y} = u\frac{\delta N}{\delta x} + N\frac{\delta u}{\delta x},$$

or

$$u\left(\frac{\delta M}{\delta y} - \frac{\delta N}{\delta x}\right) = N\frac{\delta u}{\delta x} - M\frac{\delta u}{\delta y} \qquad (3)$$

Also, by reversing the argument above, it can be seen that if *u* satisfies (3) then *u* is an integrating factor for (1). Thus, the problem of solving the ODE (1) has been “reduced” to the problem of obtaining a particular solution of the PDE (3). However at this point we do not have any method for attacking equations such as (3) so we shall restrict *u* to be a function of only one variable.

First, let *u* be a function of *x* alone. Then $\partial u/\partial y = 0$ and $\partial u/\partial x = du/dx$ and (3) reduces to

$$u\left(\frac{\delta M}{\delta y} - \frac{\delta N}{\delta x}\right) = N\frac{du}{dx}$$

or

$$\frac{1}{N}\left(\frac{\delta M}{\delta y}-\frac{\delta N}{\delta x}\right)dx=\frac{du}{u} \qquad (4)$$

If the left side of (4) is a function of $x$ alone, then we can determine $u$ at once, i.e. if

$$\frac{1}{N}\left(\frac{\delta M}{\delta y}-\frac{\delta N}{\delta x}\right)=f(x),$$

then the desired integrating factor is

$$u=e^{\int f(x)dx}.$$

Similarly, assuming that $u$ is a function of $y$ alone, i.e. if

$$\frac{1}{M}\left(\frac{\delta M}{\delta y}-\frac{\delta N}{\delta x}\right)=g(y),$$

then an integrating factor for (1) is

$$u=e^{-\int g(y)dy}.$$

**Summary:**

(a) If $\frac{1}{N}\left(\frac{\delta M}{\delta y}-\frac{\delta N}{\delta x}\right)=f(x)$, a function of $x$ alone, then $e^{\int f(x)dx}$ is an integrating factor for (1).

(b) If $\frac{1}{M}\left(\frac{\delta M}{\delta y}-\frac{\delta N}{\delta x}\right)=g(y)$, a function of $y$ alone, then $e^{-\int g(y)dy}$ is an integrating factor for (1).

(c) If neither of (a) or (b) is satisfied, we can only conclude that the equation does not have an integrating factor that is a function of $x$ or $y$ alone.

**Example #4)** $(xy+1)dx+x(x+4y-2)dy=0$

Solution:

$$(xy+1)dx+x(x+4y-2)dy=0$$

$$M=xy+1, \qquad N=x^2+4xy-2x$$

$$\frac{\delta M}{\delta y}=x, \qquad \frac{\delta N}{\delta x}=2x+4y-2$$

$$\frac{\delta M}{\delta y}-\frac{\delta N}{\delta x}=x+4y-2$$

$$\frac{1}{N}\left(\frac{\delta M}{\delta y}-\frac{\delta N}{\delta x}\right)=\frac{x+4y-2}{x(x+4y-2)}=\frac{1}{x}\Rightarrow \text{I.F. is } e^{\int\frac{dx}{x}}=e^{\ln x}=x$$

$$\left(x^2y+x\right)dx+\left(x^3+4x^2y-2x^2\right)dy=0$$

$$\left(x^2ydx+x^3dy\right)+xdx+\left(4x^2y-2x^2\right)dy=0$$

$$x^2(ydx+xdy)+xdx+2x^2(2y-1)dy=0$$

$$(ydx+xdy)+\frac{dx}{x}+2(2y-1)dy=0$$

$$xy+\ln|x|+2y^2-2y=c$$

**5.3 Substitution Suggested by the Equation**

**Exercises**

**2)** $\sin y(x+\sin y)dx + 2x^2\cos y dy = 0$

Solution:

Let $w = \sin y$. Then $dw = \cos y dy$.

$\sin y(x+\sin y)dx + 2x^2\cos y dy = 0$

$w(x+w)dx + 2x^2 dw = 0$

$(wx + w^2)dx + 2x^2 dw = 0$

Let $w = vx$. Then $dw = vdx + xdv$.

$(vx^2 + v^2x^2)dx + 2x^2(vdx + xdv) = 0$

$x^2(v+v^2)dx + 2x^2 vdx + 2x^3 dv = 0$

$x^2(3v+v^2)dx + 2x^3 dv = 0$

$$\frac{dx}{x} + \frac{2}{3v+v^2}dv = 0$$

Now,

$$\frac{2}{3v+v^2} = \frac{2}{v(3+v)} = \frac{A}{v} + \frac{B}{3+v}$$

$$A = \left.\frac{2}{3+v}\right]_{v=0} = \frac{2}{3}$$

$$B = \left.\frac{2}{v}\right]_{v=-3} = -\frac{2}{3}$$

So,

$$\frac{dx}{x} + \frac{2}{3v}dv - \frac{2}{3(3+v)}dv = 0$$

$$\ln|x| + \frac{2}{3}\ln\left|\frac{w}{x}\right| - \frac{2}{3}\ln\left|3+\frac{w}{x}\right| = \ln c_1$$

$x^3w^2 = c(3x+w)^2$

$x^3\sin^2 y = c(3x+\sin y)^2$

Find the particular solution required.

**22)** $4(3x+y-2)dx - (3x+y)dy = 0$; when $x = 1, y = 0$.

Solution:

Let $w = 3x + y$. Then $dw = 3dx + dy \Rightarrow dy = dw - 3dx$.

$$4(3x+y-2)dx-(3x+y)dy=0$$
$$4(w-2)dx-wdy=0$$
$$4(w-2)dx-w(dw-3dx)=0$$
$$(4w-8+3w)dx-wdw=0$$
$$(7w-8)dx-wdw=0$$

$$\begin{array}{r} \frac{1}{7} \\ 7w-8\overline{)w} \\ -\left(w-\frac{8}{7}\right) \\ \hline \frac{8}{7} \end{array}$$

$$dx-\frac{w}{7w-8}dw=0$$
$$dx-\frac{1}{7}dw-\frac{8}{7}\frac{1}{7w-8}dw=0$$
$$x-\frac{1}{7}w-\frac{8}{49}\ln|7w-8|=c_1$$
$$49x-7w-8\ln|7w-8|=c$$
$$49x-7(3x+y)-8\ln|7(3x+y)-8|=c$$
$$28x-7y-8\ln|21x+7y-8|=c$$

when $x=1,\ y=0$:

$$28-0-8\ln 13=c\Rightarrow c=28-8\ln 13$$

Thus, the particular solution is

$$28x-7y-28=8\ln\frac{21x+7y-8}{13}$$

### 5.4 Bernoulli's Equation

General Form of Bernoulli's Equation:

$$y'+P(x)y=Q(x)y^n \qquad (1)$$

Equation (1) may be put in the form

$$y^{-n}dy+Py^{-n+1}dx=Qdx\,. \qquad (2)$$

But the differential of $y^{-n+1}$ is $(1-n)y^{-n}dy$, so (2) may be simplified by letting

$$y^{-n+1}=z\,,$$

from which

$$(1-n)y^{-n}dy=dz\,.$$

Thus the equation in $z$ and $x$ is

$$dz+(1-n)Pzdx=(1-n)Qdx\,,$$

a linear equation in standard form.

**Exercises**

**4)** $y' = y - xy^3 e^{-2x}$

Solution:

$$y' = y - xy^3 e^{-2x}$$
$$y' - y = -xe^{-2x} y^3 \qquad \text{B.E.}$$
$$y^{-3}dy - y^{-2}dx = -xe^{-2x}dx$$
Let $z = y^{-2}$. Then $dz = -2y^{-3}dy$.
$$dz + 2zdx = 2xe^{-2x}dx \Rightarrow \text{I.F. } e^{\int 2dx} = e^{2x}$$
$$e^{2x}dz + 2e^{2x}zdx = 2xdx$$
$$d\left(ze^{2x}\right) = 2xdx$$
$$ze^{2x} = x^2 + c$$
$$y^{-2}e^{2x} = x^2 + c$$
$$e^{2x} = y^2\left(x^2 + c\right)$$

**6)** $xydx + \left(x^2 - 3y\right)dy = 0$

Solution:

Method A:
$$xydx + \left(x^2 - 3y\right)dy = 0$$
$$xydx + x^2dy - 3ydy = 0$$
$$x\left(ydx + xdy\right) - 3ydy = 0$$
$$x\ d\left(xy\right) - 3ydy = 0$$
$$x^{k+1}y^n d\left(xy\right) - 3x^k y^{n+1} = 0$$
$k + 1 = n$ and $k = 0 \Rightarrow n = 1$
$$xy\ d\left(xy\right) - 3y^2dy = 0$$
$$\frac{x^2y^2}{2} - y^3 = c_1$$
$$x^2y^2 - 2y^3 = c$$

Method B:
$$xydx + \left(x^2 - 3y\right)dy = 0$$
$$xydx + x^2dy - 3ydy = 0$$
$$\frac{dx}{dy} + \frac{1}{y}x = \frac{3}{x}$$
$$\frac{dx}{dy} + \frac{1}{y}x = 3x^{-1} \qquad \text{B.E.}$$
$$xdx + x^2y^{-1}dy = 3dy$$
Let $z = x^2$. Then $dz = 2xdx$
$$dz + 2zy^{-1}dy = 6dy \Rightarrow \text{I.F. } e^{\int \frac{2}{y}dy} = y^2$$
$$y^2dz + 2zydy = 6y^2dy$$
$$y^2z = 2y^3 + c$$
$$y^2x^2 = 2y^3 + c$$

**8)** $y' = 1 + 6xe^{x-y}$

Solution:

$$y' = 1 + 6xe^{x-y}$$
$$y' = 1 + \frac{6xe^x}{e^y}$$
$$y' = e^y + 6xe^x$$

**5.5 Coefficients Linear in Two Variables**

Consider the equation

$$(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0 \qquad (1)$$

in which the *a*'s, *b*'s, and *c*'s are constants. When $c_1$ and $c_2$ are 0, then the coefficients in (1) are each homogeneous and of degree 1 in *x* and *y*. We shall attempt to reduce (1) to that situation.

Consider the lines

$$\begin{aligned} a_1x+b_1y+c_1&=0 \\ a_2x+b_2y+c_2&=0 \end{aligned} \qquad (2)$$

They may be parallel or they may intersect. There will not be two lines if $a_1$ and $b_1$ are zero or if $a_2$ and $b_2$ are zero, but then (1) will be linear in one of its variables.

If the lines intersect, let the point of intersection be $(h,k)$. Then the translation

$$\begin{aligned} x&=u+h \\ y&=v+k \end{aligned} \qquad (3)$$

will change the equations (2) into equations of lines through the origin of the *uv*-coordinate system, namely,

$$\begin{aligned} a_1u+b_1v&=0 \\ a_2u+b_2v&=0 \end{aligned} \qquad (4)$$

Therefore, since $dx=du$ and $dy=dv$, the change of variables will transform the d.e. (1) into

$$(a_1u+b_1v)du+(a_2u+b_2v)dv=0 \qquad (5)$$

an equation we can solve.

If the lines (2) do not intersect, there exists a constant *k* s.t.

$$a_2x+b_2y=k(a_1x+b_1y),$$

so that (1) appears in the form

$$(a_1x+b_1y+c_1)dx+\left[k(a_1x+b_1y)+c_2\right]dy=0 \qquad (6)$$

and the substitution $w=a_1x+b_1y$ will lead to a separable equation.

**Exercises**

**2)** $(x-4y-9)dx+(4x+y-2)dy=0$

Solution:

Find the intersection of the lines:

$$\begin{aligned}
x-4y-9=0 &\longrightarrow \quad x-4y-9=0\\
4x+y-2=0 &\xrightarrow{\times 4} \underline{16x+4y-8=0}\\
& \quad 17x \qquad -17=0 \Rightarrow x=1
\end{aligned}$$

$$1-4y-9=0 \Rightarrow -8-4y=0 \Rightarrow y=-2$$

Then set

$$x=u+1$$
$$y=v-2$$

to get

$$\big(u+1-4(v-2)-9\big)dx+\big(4(u+1)+v-2-2\big)dy=0$$
$$(u-4v)du+(4u+v)dv=0$$

which has homogeneous coefficients and degree 1 in $u$ and $v$. Let $u=vz$:

$$(vz-4v)(vdz+zdv)+(4vz+v)dv=0$$
$$\left(v^2z-4v^2\right)dz+\left(vz^2-4vz\right)dv+4vzdv+vdv=0$$
$$v^2(z-4)dz+v\left(z^2+1\right)dv=0$$
$$\frac{z-4}{z^2+1}dz+\frac{1}{v}dv=0$$
$$\frac{1}{2}\ln\left(z^2+1\right)-4\arctan z+\ln|v|=c$$
$$\ln|v|\sqrt{\frac{u^2}{v^2}+1}-4\arctan\left(\frac{u}{v}\right)=c$$
$$\ln\sqrt{u^2+v^2}-4\arctan\left(\frac{u}{v}\right)=c$$
$$\ln\sqrt{(x-1)^2+(y+2)^2}-4\arctan\left(\frac{x-1}{y+2}\right)=c$$

**8)** $(6x-3y+2)dx-(2x-y-1)dy=0$

Solution:

The lines

$$6x-3y+2=0$$
$$2x-y-1=0$$

are parallel so we let $v=2x-y$ and $dv=2dx-dy$ or $2dx=dv+dy$ to get

$$\frac{1}{2}(3v+2)(dv+dy)-(v-1)dy=0$$
$$(3v+2)(dv+dy)-2(v-1)dy=0$$
$$(3v+2)dv+(3v+2-2v+2)dy=0$$
$$(3v+2)dv+(v+4)dy=0$$
$$\frac{3v+2}{v+4}dv+dy=0$$
$$\left(3-\frac{10}{v+4}\right)dv+dy=0$$
$$3v-10\ln|v+4|+y=c$$
$$3(2x-y)-10\ln|2x-y+4|+y=c_1$$
$$6x-2y-10\ln|2x-y+4|=c_1$$
$$3x-y-5\ln|2x-y+4|=c$$

$$\begin{array}{r} 3 \\ v+4\,\overline{)\,3v+2} \\ -(3v+12) \\ \hline -10 \end{array}$$

**5.6 Solutions Involving Nonelementary Integrals**

Some nonelementary integrals:

| | | |
|---|---|---|
| $\int e^{-x^2}dx$ | $\int \frac{e^{-x}}{x}dx$ | $\int x\tan x dx$ |
| $\int \sin x^2 dx$ | $\int \frac{\sin x}{x}dx$ | $\int \frac{dx}{\ln x}$ |
| $\int \cos x^2 dx$ | $\int \frac{\cos x}{x}dx$ | $\int \frac{dx}{\sqrt{1-x^3}}$ |