Math 275 Elementary Differential Equations

Chapter 4 Elementary Applications

4.1 Velocity of Escape from the Earth

Consider the problem of determining the velocity of a particle projected in a radial direction outward from the earth and acted upon by only one force, the gravitational attraction of the earth. Assume an initial velocity in a radial direction so that the motion of the particle takes place entirely on a line through the center of the earth.

Let r be the variable distance of the particle to the center of the earth
- R the radius of the earth
- t time
- v velocity of the particle
- a acceleration
- k proportionality constant

Then Newton's law of gravitation states

$$a = \frac{dv}{dt} = -\frac{k}{r^2},$$

i.e. the acceleration of the particle is inversely proportional to the square of the distance from the particle to the center of the earth. Note that acceleration is negative because velocity is decreasing. Hence the constant k is positive.

When $r = R$, then $a = -g$, the acceleration of gravity at the surface of the earth. Thus,

$$-g = -\frac{k}{R^2},$$

from which

$$a = -\frac{gR^2}{r^2}.$$

We wish to express the acceleration in terms of the velocity and the distance. We have $a = \frac{dv}{dt}$ and $v = \frac{dr}{dt}$, hence

$$a = \frac{dv}{dt} = \frac{dr}{dt}\frac{dv}{dr} = v\frac{dv}{dr},$$

so the DE for the velocity is

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2}. \qquad (1)$$

We can separate the variables to get the set of solutions

$$v^2 = \frac{2gR^2}{r} + C.$$

Suppose that the particle leaves the earth's surface with the velocity v_0. Then $v = v_0$ when $r = R$, and thus the constant C is

$$C = v_0^2 - 2gR.$$

Thus a particle projected in a radial direction outward from the earth's surface with an initial velocity v_0 will travel with a velocity v given by the equation

$$v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR. \qquad (2)$$

Next, we want to determine whether the particle will escape from the earth. At the earth's surface, at $r = R$, the velocity $v = v_0$ is positive. By (2), the velocity will remain positive if, and only if,

$$v_0^2 - 2gR \geq 0. \qquad (3)$$

If the inequality (3) is satisfied, the velocity given by (2) will remain positive because it cannot vanish, is continuous, and is positive at $r = R$. On the other hand, if (3) is not satisfied, then $v_0^2 - 2gR < 0$, and there will be a critical value of r for which the right member of (2) is zero, i.e., the particle would stop, the velocity would change from positive to negative, and the particle would return to the earth.

A particle projected from the earth with a velocity v_0 such that

$$v_0 \geq \sqrt{2gR}$$

will escape from the earth. Hence the minimum such velocity of projection,

$$v_e = \sqrt{2gR},$$

is called the **velocity of escape**.

The radius of the earth is approximately $R = 3960$ miles. The acceleration of gravity at the surface of the earth is approximately $g = 32.16$ ft/sec^2 or 6.09×10^{-3} mi/sec^2. For the earth, the velocity of escape is $v_e = 6.95$ mi/sec.

Note: The formula $v_e = \sqrt{2gR}$ applies for other members of the solar system as well.

4.2 Newton's Law of Cooling

Suppose a thermometer, which has been at the reading 70°F inside a house, is placed outside where the air temperature is 10°F. Three minutes later the temperature reading is 25°F. What is the temperature reading at any time?

Let u be the temperature (in $^\circ$F) of the thermometer at time t (in min)
t the time measured from the instant the thermometer is brought outside

Given: when $t = 0$, $u = 70$ and when $t = 3$, $u = 25$.

Newton's Law: The temperature of a body changes at a rate that is proportional to the difference in temperature between the outside medium and the body itself.

Applying Newton's Law, then u has to be determined from

$$\frac{du}{dt} = -k(u-10), \qquad (1)$$

and the initial conditions that

$$\text{when } t = 0,\ u = 70 \qquad (2)$$

and

$$\text{when } t = 3,\ u = 25. \qquad (3)$$

From (1) we get

$$u = 10 + Ce^{-kt}.$$

Substituting (2), we get $C = 60$ so we have

$$u = 10 + 60e^{-kt}. \qquad (4)$$

Using (3) we get

$$25 = 10 + 60e^{-3k},$$

from which $e^{-3k} = \frac{1}{4}$, so $k = \frac{1}{3}\ln 4$.

Thus the temperature is given by

$$u = 10 + 60e^{-\frac{1}{3}t\ln 4}. \qquad (5)$$

Using $\ln 4 = 1.39$, we may write (5) as

$$u = 10 + 60e^{-0.46t}. \qquad (6)$$

4.3. Simple Chemical Conversion

In certain chemical reactions in which a substance A is being converted into another substance, the time rate of change of the amount x of unconverted substance is proportional to x.

Let the amount of unconverted substance be known at some specified time, i.e. let $x = x_0$ at $t = 0$. Then the amount x at any time $t > 0$ is determined by the DE

$$\frac{dx}{dt} = -kx \qquad (1)$$

and the condition that $x = x_0$ when $t = 0$. Since the amount x is decreasing as time increases, the constant of proportionality is taken to be $(-k)$.

From (1) it follows that

$$x = Ce^{-kt},$$

and substituting the given condition we get

$$x = x_0 e^{-kt}. \qquad (2)$$

We need another condition to determine *k*, so suppose we know that at the end of half a minute $(t = 30 \text{ sec})$, two-thirds of the original amount x_0 has already been converted. Let us find how much unconverted substance remains at $t = 60$ sec.

When two-thirds of the substance has been converted, one-third remains unconverted. Hence $x = \frac{1}{3}x_0$ when $t = 30$. So (2) gives

$$\tfrac{1}{3}x_0 = x_0 e^{-30k} \Rightarrow k = \tfrac{1}{30}\ln 3.$$

Thus the amount of unconverted substance is given by

$$x = x_0 e^{-\frac{1}{30}t\ln 3}. \qquad (3)$$

At $t = 60$,

$$x = x_0 e^{-2\ln 3} = x_0 (3)^{-2} = \tfrac{1}{9}x_0.$$

4.4 Logistic Growth and the Price of Commodities

Logistic Growth

Assume that the average birthrate per individual is a positive constant and the average death rate per individual is proportional to the population. If we let $x(t)$ represent the population at time *t*, then we get the DE

$$\frac{1}{x}\frac{dx}{dt} = b - ax, \qquad (1)$$

where *b* and *a* are positive constants. (1) is commonly called the **logistic equation** and the growth of population determined by it is called **logistic growth**.

Solving (1) by separating the variables, we get

$$\frac{dx}{x(b-ax)} = dt$$

$$\left(\frac{1}{x} + \frac{a}{b-ax}\right)dx = dt$$

$$\ln\left|\frac{x}{b-ax}\right| = bt + c$$

or

$$\left|\frac{x}{b-ax}\right| = e^c e^{bt}. \qquad (2)$$

Assuming that at $t = 0$ the population is $x_0 > 0$, we can write (2) as

$$\frac{x}{b-ax} = \frac{x_0}{b-ax_0} e^{bt},$$

and solving for *x* we get

$$x(t) = \frac{bx_0 e^{bt}}{b - ax_0 + ax_0 e^{bt}}. \qquad (3)$$

Note that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{bx_0 e^{bt}}{b - ax_0 + ax_0 e^{bt}}$$
$$= \lim_{t\to\infty} \frac{b^2 x_0 e^{bt}}{abx_0 e^{bt}} = \frac{b}{a}.$$

Note that (1) will dictate growth or decline in the population, depending on whether the initial population is less than or greater than b/a.

Price of Commodities

Assume that the price *P*, the supply *S*, and the demand *D* of a commodity are functions of time and that the rate of change of the price is proportional to the difference between the demand and the supply, i.e.

$$\frac{dP}{dt} = k(D - S). \qquad (4)$$

Assume that the constant *k* is positive so that price will increase if the demand exceeds the supply.

Different models of the commodity market will result, depending on the nature of the demand and supply functions that are indicated. For example, if we assume that

$$D = c - dP \quad \text{and} \quad S = a + bP, \qquad (5)$$

where *a*, *b*, *c*, and *d* are positive constants, we obtain a DE

$$\frac{dP}{dt} = k\left[(c-a) - (d+b)P\right] \qquad (6)$$

that is linear in *P*. The assumptions (5) reflect the tendency for the demand to decrease as the price increases and the tendency for the supply to increase as the price increases, both reasonable assumptions for many commodities. We should also assume that $0 < P < c/d$, so that *D* is not negative.

Equation (6) may be written

$$\frac{dP}{dt} + k(d+b)P = k(c-a), \qquad (7)$$

and solved by multiplying by the integrating factor $e^{k(d+b)t}$ and integrating to obtain

$$P(t) = c_1 e^{-k(d+b)t} + \frac{c-a}{d+b}.$$

If the price at $t = 0$ is $P = P_0$, we have

$$c_1 = P_0 - \frac{c-a}{d+b},$$

so that

$$P(t) = \left(P_0 - \frac{c-a}{d+b}\right)e^{-k(d+b)t} + \frac{c-a}{d+b}. \qquad (8)$$

Equation (8) shows that under the assumptions of (4) and (5) the price will stabilize at a value $\frac{c-a}{d+b}$ as *t* becomes large.