Math 275 Elementary Differential Equations

Chapter 3 Numerical Methods

3.2 Euler's Method

To obtain a solution of the DE

$$y' = y^2 - x^2 \qquad (1)$$

for which $y = 1$ when $x = 0$ and to approximate the solution $y = y(x)$ in the interval $0 \le x \le \frac{1}{2}$.

Equation (1) may be written as

$$dy = \left(y^2 - x^2\right)dx. \qquad (2)$$

We know the value of y at $x = 0$; to compute y for $0 \le x \le \frac{1}{2}$. Choose $\Delta x = 0.1$; then dy can be computed from

$$dy = \left(y^2 - x^2\right)\Delta x$$

and we get $dy = (1-0)(0.1) = 0.1$. Thus for $x = 0 + 0.1$ the approximate value of y is $1 + 0.1$. Choose $\Delta x = 0.1$ again. Then

$$dy = \left[(1.1)^2 - (0.1)^2\right](0.1) = 0.12.$$

Hence at $x = 0.2$, y is approximately 1.22. The following table shows the computations:

x	y	y^2	x^2	$y^2 - x^2$	dy
0.0	1.000	1.000	0.000	1.000	0.100
0.1	1.100	1.210	0.010	1.200	0.120
0.2	1.220	1.488	0.040	1.448	0.145
0.3	1.365	1.863	0.090	1.773	0.177
0.4	1.542	2.378	0.160	2.218	0.222
0.5	1.764				

The increment need not be constant throughout the interval but computations are simpler if equal increments are used. The next table shows the values of y obtained when $\Delta x = 0.05$.

x	y	y^2	x^2	$y^2 - x^2$	dy
0.00	1.000	1.000	0.000	1.000	0.050
0.05	1.050	1.102	0.002	1.100	0.055
0.10	1.105	1.221	0.010	1.211	0.061
0.15	1.166	1.359	0.022	1.336	0.067
0.20	1.232	1.519	0.040	1.479	0.074
0.25	1.306	1.706	0.062	1.644	0.082
0.30	1.388	1.928	0.090	1.838	0.092
0.35	1.480	2.192	0.122	2.069	0.103
0.40	1.584	2.508	0.160	2.348	0.117
0.45	1.701	2.894	0.202	2.692	0.135
0.50	1.836				

The following shows the values of y obtained using the increments $\Delta x = 0.1,\ 0.05,\ 0.01$:

When x	$\Delta x = 0.1$ y	$\Delta x = 0.05$ y	$\Delta x = 0.01$ y	Correct y
0.0	1.000	1.000	1.000	1.000
0.1	1.100	1.105	1.110	1.111
0.2	1.220	1.232	1.244	1.247
0.3	1.365	1.388	1.411	1.417
0.4	1.542	1.584	1.625	1.637
0.5	1.764	1.836	1.911	1.934

Euler's Method

To solve

$$\frac{dy}{dx} = f(x,y)\text{; when } x = x_0, \quad y = y_0, \tag{3}$$

perform the following sequence of approximations

$$\begin{aligned} x_{k+1} &= x_k + h \\ y_{k+1} &= y_k + hf(x_k, y_k), \end{aligned} \tag{4}$$

for $k = 0,1,2,\ldots$. Note that $h = \Delta x$.

3.3 A Modification of Euler's Method

Euler's method uses the slope $f(x_k, y_k)$ to get the new approximation y_{k+1}, i.e. the slope is calculated at the left endpoint of the interval $x_k \le x \le x_k + h$. A modification of the method is to use the ***midpoint*** of the interval.

Starting at the initial point (x_0, y_0) and using Euler's method to determine (x_1, y_1), we start over again at the initial point (x_0, y_0). This time, however, we use Euler's method with increment size $2h$ and use the value of the slope at the point (x_1, y_1), a point that lies at the midpoint of the new interval

$$x_0 \le x \le x_0 + 2h.$$

Thus, the formulas for the modified Euler's method are

$$\begin{aligned} x_1 &= x_0 + h \\ y_1 &= y_0 + hf(x_0, y_0) \end{aligned}$$

and

$$\begin{aligned} x_{k+2} &= x_k + 2h, \quad k \ge 0, \\ y_{k+2} &= y_k + 2hf(x_{k+1}, y_{k+1}), \quad k \ge 0. \end{aligned}$$

We get the following:

When x	$\Delta x = 0.1$ y	$\Delta x = 0.05$ y	$\Delta x = 0.01$ y	Correct y
0.0	1.000	1.000	1.000	1.000
0.1	1.100	1.100	1.111	1.111
0.2	1.240	1.245	1.247	1.247
0.3	1.400	1.414	1.417	1.417
0.4	1.614	1.631	1.637	1.637
0.5	1.888	1.922	1.933	1.934

3.4 A Method of Successive Approximation

Now let us attack the same problem

$$y' = y^2 - x^2; \quad x = 0, y = 1, \tag{1}$$

with y desired in the interval $0 \le x \le \frac{1}{2}$, by the method suggested in the discussion of the existence theorem in Chapter 13. Applying the statements to be made in that discussion, we conclude that the desired solution is $y = y(x)$, where

$$y(x) = \lim_{n \to \infty} y_n(x)$$

and the sequence of functions $y_n(x)$ is given by $y_0(x) = 1$, and for $n \ge 1$,

$$y_n(x) = 1 + \int_0^x \left[y_{n-1}^2(t) - t^2 \right] dt. \tag{2}$$

For the given problem,

$$\begin{aligned} y_1(x) &= 1 + \int_0^x (1 - t^2)\, dt, \\ &= 1 + x - \tfrac{1}{3} x^3. \end{aligned}$$

Next we get a second approximation, finding y_2 from y_1 using (2) and we get

$$\begin{aligned} y_2(x) &= 1 + \int_0^x \left[\left(1 + t - \tfrac{1}{3} t^3\right)^2 - t^2 \right] dt, \\ &= 1 + x + x^2 - \tfrac{1}{6} x^4 - \tfrac{2}{15} x^5 + \tfrac{1}{63} x^7. \end{aligned}$$

Then $y_3(x), y_4(x), \ldots,$ can be obtained in a similar manner, each from the preceding element of the sequence $y_n(x)$.

x	$y_1(x)$	$y_2(x)$	$y_3(x)$	$y(x)$
0.0	1.00	1.00	1.00	1.00
0.1	1.10	1.11	1.11	1.11
0.2	1.20	1.24	1.25	1.25
0.3	1.29	1.39	1.41	1.42
0.4	1.38	1.56	1.62	1.64
0.5	1.46	1.74	1.87	1.93

3.5 An Improvement on the Method of Successive Approximation

The initial value problem we are solving is

$$y' = y^2 - x^2; \quad x = 0, y = 1,$$

and it tells us that at $x = 0$, the slope is $y' = 1$.

By blindly following the suggestion from Chapter 13, we started out with $y_0(x) = 1$, a line that does not have the correct slope at $x = 0$. We therefore alter our initial approximation by choosing y_0 to have the correct slope at $(0,1)$, thus

$$y_0(x) = 1 + x$$

and proceed to compute $y_1(x), y_2(x), \ldots,$ as before. The successive stages of approximation to $y(x)$ now become

$$\begin{aligned} y_1(x) &= 1 + \int_0^x \left[(1+t)^2 - t^2 \right] dt \\ &= 1 + x + x^2; \\ y_2(x) &= 1 + \int_0^x \left[\left(1 + t + t^2\right)^2 - t^2 \right] dt \\ &= 1 + x + x^2 + \tfrac{2}{3}x^3 + \tfrac{1}{2}x^4 + \tfrac{1}{5}x^5; \end{aligned}$$

and so on. We have the following table of values:

x	$y_1(x)$	$y_2(x)$	$y_3(x)$	$y(x)$
0.0	1.00	1.00	1.00	1.00
0.1	1.11	1.11	1.11	1.11
0.2	1.24	1.25	1.25	1.25
0.3	1.39	1.41	1.42	1.42
0.4	1.56	1.62	1.64	1.64
0.5	1.75	1.87	1.92	1.93

3.6 The Use of Taylor's Theorem

Consider the initial value problem

$$y' = F(x,y); \quad x = x_0, \ \ y = y_0, \tag{1}$$

we may be able to compute successive derivatives of the solution $y = y(x)$ at $x = x_0$ by using (1). We adopt the notation

$$y_0' = y'(x_0), \quad y_0'' = y''(x_0), \quad \ldots \ \ y_0^{(n)} = y^{(n)}(x_0),$$

and recall that Taylor's theorem suggests the approximation formula

$$y \approx y_0 + y_0'(x - x_0) + \frac{y_0''}{2!}(x - x_0)^2 + \cdots + \frac{y_0^{(n)}}{n!}(x - x_0)^n. \tag{2}$$

We may be able to estimate the error on our calculation by examining the value of the remainder term in Taylor's theorem which takes the form

$$\frac{y^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \tag{3}$$

where c is some number between x and x_0.

For our example

$$y' = y^2 - x^2; \quad x = 0, y = 1, \tag{4}$$

we have

$$\begin{aligned} y'' &= 2yy' - 2x, \\ y''' &= 2yy'' + 2(y')^2 - 2, \\ y^{(4)} &= 2yy''' + 6y'y'', \\ y^{(5)} &= 2yy^{(4)} + 8y'y''' + 6(y'')^2. \end{aligned} \tag{5}$$

Thus we can obtain the values

$$y_0 = 1, \ \ y_0' = 1, \ \ y_0'' = 2, \ \ y_0''' = 4, \ \ y_0^{(4)} = 20, \quad \text{and} \ \ y_0^{(5)} = 96.$$

Equation (2) thus becomes

$$y \approx 1 + x + x^2 + \tfrac{2}{3}x^3 + \tfrac{5}{6}x^4 . \tag{6}$$

x	y	Correct y
0.0	1.00	1.00
0.1	1.11	1.11
0.2	1.25	1.25
0.3	1.41	1.42
0.4	1.62	1.64
0.5	1.89	1.93

3.7 The Runge-Kutta Method

Consider the initial value problem

$$y' = F(x,y); \quad \text{when } x = x_n, \quad y = y_n, \tag{1}$$

and adopt the notation

$$F_n = F(x_n, y_n). \tag{2}$$

Let us begin by considering the tangent line to the solution curve at the point (x_n, y_n). The equation of this line is given by

$$y = y_n + F_n(x - x_n). \tag{3}$$

The value of y for this tangent line at $x = x_n + h$ is $y = y_n + hF_n$. If we define $K_1 = F_n$ and compute F at the point $(x_n + h, y_n + hK_1)$, we obtain $K_2 = F(x_n + h, y_n + hK_1)$. Thus K_1 and K_2 represent the values of y' at two points, the two endpoints of a segment of a tangent line. If we consider the arithmetic mean of these values of y', namely, $\frac{1}{2}(K_1 + K_2)$, and replace the tangent line with a new line through (x_n, y_n) having this slope, we obtain

$$y = y_n + \tfrac{1}{2}(K_1 + K_2)(x - x_n).$$

For $x = x + h$, this line has a point whose y coordinate is

$$y = y_n + \tfrac{h}{2}(K_1 + K_2), \tag{4}$$

where

$$K_1 = F_n \tag{5}$$

and

$$K_2 = F(x_n + h, y_n + hK_1). \tag{6}$$