Math 275 Elementary Differential Equations

Chapter 2 Equations of Order One

In this chapter we study several methods for solving first-order differential equations

$$\frac{dy}{dx} = f(x,y)$$

which we will write in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 .$$

We shall focus on 4 types of equations:

1. separable variables
2. homogeneous equations
3. exact equations
4. linear equations

2.1 Separation of Variables

Sometimes differential equations $M(x,y)\,dx + N(x,y)\,dy = 0$ are so simple that they can be written in the form

$$A(x)\,dx + B(y)\,dy = 0 \qquad (1)$$

i.e. the variables can be separated.

Examples

1. Solve $(1+x)\,dy - y\,dx = 0$.

Solution:

Divide by $(1+x)\,y$:

$$\frac{dy}{y} = \frac{dx}{1+x}$$

Integrate:

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln|y| = \ln|1+x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{c_1} e^{\ln|1+x|} = e^{c_1}\,|1+x| = \pm e^{c_1}\,(1+x)$$

$$y = c(1+x)$$

Note: Had we used $\ln c$ instead of c, we would have gotten

$$\ln|y| = \ln|1+x| + \ln c$$

$$\ln|y| = \ln|c(1+x)|$$

$$y = c(1+x)$$

2. Solve the IVP:

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = 3$$

Solution:

$$ydy = -xdx$$

$$\int ydy = -\int xdx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

$$y^2 = -x^2 + 2c_1$$

$x^2 + y^2 = c^2$ family of concentric circles

when $x = 4,\ y = 3$:

$$4^2 + 3^2 = c^2 \Rightarrow c^2 = 25$$

Thus, the IVP determines $\boxed{x^2 + y^2 = 25}$.

3. Solve $xe^{-y} \sin x dx - ydy = 0$.

Solution:

Multiply by e^y:

$$x \sin x dx = ye^y dy$$

$$\int x \sin x dx = \int ye^y dy$$

Integrate by parts (tabular integration):

x	$\sin x$		y	e^y
1	$\cos x$		1	e^y
0	$-\sin x$		0	e^y

$$\boxed{-x\cos x + \sin x = ye^y - e^y + c}$$

Technique:

a) Equation is separable if it can be put in the form $\dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$.

b) Put all terms involving x on one side of the equation and all terms involving y on the other side.

c) Integrate both sides of the equation.

2.2 Homogeneous Functions

Polynomials in which all terms are of the same degree are called **homogeneous** polynomials.

Def The function $f(x,y)$ is said to be **homogeneous of degree *k* in *x* and *y*** if and only if

$$f(\lambda x, \lambda y) = \lambda^k f(x,y).$$

Examples Determine if the given function is homogeneous. If it is, give the degree.

1. $f(x,y) = x^2 - 3xy + 5y^2$

Solution:

$$f(\lambda x, \lambda y) = \lambda^2 x^2 - 3\lambda^2 xy + 5\lambda^2 y^2$$
$$= \lambda^2 \left(x^2 - 3xy + 5y^2\right) = \lambda^2 f(x,y)$$

Thus, f is homogeneous of degree 2.

2. $f(x,y) = \sqrt[3]{x^2 + y^2}$

Solution:

$$f(\lambda x, \lambda y) = \sqrt[3]{\lambda^2 x^2 + \lambda^2 y^2} = \sqrt[3]{\lambda^2 \left(x^2 + y^2\right)} = \lambda^{2/3}\sqrt[3]{x^2 + y^2} = \lambda^{2/3} f(x,y)$$

Thus, f is homogeneous of degree $2/3$.

3. $f(x,y) = x^3 + y^3 + 1$

Solution:

$$f(\lambda x, \lambda y) = \lambda^3 x^3 + \lambda^3 y^3 + 1 \neq \lambda^3 f(x,y)$$

Thus, f is <u>not</u> homogeneous.

4. $f(x,y) = \dfrac{x}{2y} + 6$

Solution:

$$f(\lambda x, \lambda y) = \frac{\lambda x}{2\lambda y} + 6 = \frac{x}{2y} + 6 = \lambda^0 f(x,y)$$

Thus, f is homogeneous of degree 0.

Note:

1. A constant added to a function destroys homogeneity, unless the function is homogeneous of degree 0.

2. In many instances a homogeneous function can be recognized by examining the **<u>total degree</u>** of each term.

Examples:

$$f(x,y) = \underbrace{6xy^3}_{\text{degree 4}} - \underbrace{x^2y^2}_{\text{degree 4}} \Rightarrow f \text{ homogeneous of degree 4}$$
$$f(x,y) = \underbrace{x^2}_{\text{degree 2}} - \underbrace{y}_{\text{degree 1}} \Rightarrow f \text{ is } \underline{\text{not}} \text{ homogeneous}$$

3. If $f(x,y)$ is homogeneous of degree n, notice that we can write

$$f(x,y) = x^n f\left(1, \frac{y}{x}\right) \text{ and}$$
$$f(x,y) = y^n f\left(\frac{x}{y}, 1\right),$$

where $f\left(1, \frac{y}{x}\right)$ and $f\left(\frac{x}{y}, 1\right)$ are both homogeneous of degree 0.

Example:

$f(x,y) = x^2 + 3xy + y^2$ homogeneous of degree 2

$$f(x,y) = x^2\left(1+3\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\right) = x^2 f\left(1,\frac{y}{x}\right)$$

$$f(x,y) = y^2\left(\left(\frac{x}{y}\right)^2+3\left(\frac{x}{y}\right)+1\right) = y^2 f\left(\frac{x}{y},1\right)$$

Theorem 2.1 If $M(x,y)$ and $N(x,y)$ are both homogeneous and of the dame degree, the function $M(x,y)/N(x,y)$ is homogeneous of degree 0.

Theorem 2.2 If $f(x,y)$ is homogeneous of degree 0 in x and y, $f(x,y)$ is a function of y/x alone.

Proof:

Let $y = vx$. Then the theorem states that if $f(x,y)$ is homogeneous of degree zero, $f(x,y)$ is a function of v alone. Now

$$f(x,y) = f(x,vx) = x^0 f(1,v) = f(1,v),$$

in which the x is now playing the role taken by λ in the definition and $f(x,y)$ depends on v alone.

2.3 Equations with Homogeneous Coefficients

Suppose the coefficients M and N in an equation of order one,

$$M(x,y)dx + N(x,y)dy = 0, \qquad (1)$$

are both homogeneous functions and are of the same degree in x and y. By T 2.1 and 2.2 of Section 2.2, the ratio M/N is a function of y/x alone. Hence (1) may be put in the form

$$\frac{dy}{dx} + g\left(\frac{y}{x}\right) = 0. \qquad (2)$$

This suggests the introduction of a new variable v by putting $y = vx$. Then (2) becomes

$$x\frac{dv}{dx} + v + g(v) = 0 \qquad (3)$$

in which the variables are separable. We can also use the substitution $x = vy$ to obtain a separable equation in y and v.

Examples

1. Solve $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

Solution:

Note that both M and N are homogeneous of degree 2.
Let $y = ux$. Then $dy = udx + xdu$.

$$\left(x^2+y^2\right)dx+\left(x^2-xy\right)dy=0$$

$$\left(x^2+u^2x^2\right)dx+\left(x^2-ux^2\right)\left(udx+xdu\right)=0$$

$$x^2dx+\cancel{u^2x^2dx}+ux^2dx-\cancel{u^2x^2dx}+x^3du-ux^3du=0$$

$$x^2\left(1+u\right)dx+x^3\left(1-u\right)du=0$$

$$\frac{dx}{x}+\frac{1-u}{1+u}du=0$$

$$\frac{dx}{x}+\frac{1+u-2u}{1+u}du=0$$

$$\frac{dx}{x}+\left[1-\frac{2u}{1+u}\right]du=0$$

$$\frac{dx}{x}+\left[1-\frac{2u+2-2}{1+u}\right]du=0$$

$$\frac{dx}{x}+\left[1-2-\frac{2}{1+u}\right]du=0$$

$$\frac{dx}{x}+\left[-1-\frac{2}{1+u}\right]du=0 \qquad \text{or do long division}$$

$$\ln|x|-u-2\ln|1+u|=\ln|c|$$

$$\ln|x|-\frac{y}{x}-2\ln\left|1+\frac{y}{x}\right|=\ln|c|$$

$$\frac{y}{x}=\ln\left|\frac{\left(x+y\right)^2}{cx}\right|$$

$$\boxed{\left(x+y\right)^2=cxe^{y/x}}$$

2. Solve $2x^3ydx+\left(x^4+y^4\right)dy=0$

Solution:

M and *N* are homogeneous of degree 4. Since *M* is simpler, let $x=vy$. Then $dx=vdy+ydv$.

$$2v^3y^4\left(vdy+ydv\right)+\left(v^4y^4+y^4\right)dy=0$$

$$2v^4y^4dy+2v^3y^5dv+v^4y^4dy+y^4dy=0$$

$$3v^4y^4dy+y^4dy+2v^3y^5dv=0$$

$$y^4\left(3v^4+1\right)dy+2v^3y^5dv=0$$

$$\frac{dy}{y}+\frac{2v^3}{3v^4+1}dv=0$$

$$\ln|y|+\frac{1}{6}\ln\left(3v^4+1\right)=\ln|c_1|$$

$$\left(3v^4+1\right)y^6=c$$

$$3\frac{x^4}{y^4}y^6+y^6=c$$

$$\boxed{3x^4y^2+y^6=c}$$

2.4 Exact Equations

Given an equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (1)$$

To find a function $F(x,y)$ that has for its differential the expression $Mdx + Ndy = 0$, i.e.

$$dF = Mdx + Ndy \qquad (2)$$

Then,

$$F(x,y) = c \qquad (3)$$

implicitly defines a set of solutions of (1).

We need *two things*:

(1) to find out under what conditions on *M* and *N* a function *F* exists s.t. its total differential is exactly $Mdx + Ndy$, and

(2) if those conditions are satisfied, to actually determine the function *F*. If a function *F* exists s.t. $Mdx + Ndy$ is exactly the total differential of *F*, we call equation (1) an **exact equation**.

If equation (1) is exact then there exists *F* s.t.

$$dF = Mdx + Ndy .$$

But from calculus,

$$dF = \frac{\delta F}{\delta x}dx + \frac{\delta F}{\delta y}dy ,$$

so

$$M = \frac{\delta F}{\delta x}, \qquad N = \frac{\delta F}{\delta y}$$

and

$$\frac{\delta M}{\delta y} = \frac{\delta^2 F}{\partial y \delta x}, \qquad \frac{\delta N}{\delta x} = \frac{\delta^2 F}{\partial x \delta y} .$$

Again from calculus,

$$\frac{\delta^2 F}{\partial y \delta x} = \frac{\delta^2 F}{\partial x \delta y} ,$$

provided that these partial derivatives are continuous. Thus, if (1) is an exact equation, then

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x} \qquad (4)$$

Thus, for (1) to be exact it is necessary that (4) be satisfied.

Conversely, let us show that if condition (4) is satisfied, then (1) is an exact equation. Let $\phi(x, y)$ be a function for which

$$\frac{\delta \phi}{\delta x} = M .$$

The function ϕ is the result of integrating *Mdx* with respect to *x* while holding *y* constant. Now

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\delta M}{\delta y} ,$$

hence, if (4) is satisfied, then also

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\delta N}{\delta x} . \qquad (5)$$

Integrate both sides of (5) with respect to *x* while holding *y* fixed. Then

$$\frac{\delta \phi}{\delta y} = N + B'(y) . \qquad (6)$$

Now a function *F* can be exhibited, namely,

$$F = \phi(x, y) - B(y)$$

for which

$$\begin{aligned} dF &= \frac{\delta \phi}{\delta x} dx + \frac{\delta \phi}{\delta y} dy - B'(y) dy \\ &= Mdx + \left[N + B'(y) \right] dy - B'(y) dy \\ &= Mdx + Ndy \end{aligned}$$

Hence, (1) is exact. This completes the proof of

Theorem 2.3 If M, N, $\partial M / \partial y$, and $\partial N / \partial x$ are continuous functions of *x* and *y*, then a necessary and sufficient condition that

$$Mdx + Ndy = 0 \qquad (1)$$

be an exact equation is that

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x} \qquad (4)$$

Examples

1. Solve $2xydx+\left(x^2-1\right)dy=0$

Solution:

$$M=2xy \qquad N=x^2-1$$

$$\frac{\delta M}{\delta y}=2x \qquad \frac{\delta N}{\delta x}=2x$$

$\Rightarrow$ DE is exact $\Rightarrow$ there exists F s.t

$$\frac{\delta F}{\delta x}=M=2xy \text{ and } \frac{\delta F}{\delta y}=N=x^2-1$$

$$F(x,y)=x^2y+g(y)$$

$$\frac{\delta F}{\delta y}=x^2+g'(y)=x^2-1$$

$$\Rightarrow g'(y)=-1\Rightarrow g(y)=-y$$

Thus, the solution is the family of curves $\boxed{x^2y-y=c}$.

2. Solve $\left(e^{2y}-y\cos xy\right)dx+\left(2xe^{2y}-x\cos xy+2y\right)dx=0$.

Solution:

$$\frac{\delta M}{\delta y}=2e^{2y}+xy\sin xy-\cos xy=\frac{\delta N}{\delta x}$$

$$\Rightarrow \exists F \text{ s.t. } M(x,y)=\frac{\delta F}{\delta x} \text{ and } N(x,y)=\frac{\delta F}{\delta y}.$$

For variety's sake, let us use the second:

$$\frac{\delta F}{\delta y}=2xe^{2y}-x\cos xy+2y$$

$$F(x,y)=2x\int e^{2y}dy-x\int \cos xydy+2\int ydy$$

$$F(x,y)=xe^{2y}-\sin xy+y^2+h(x)$$

$$\frac{\delta F}{\delta x}=e^{2y}-y\cos xy+h'(x)=e^{2y}-y\cos xy$$

$$\Rightarrow h'(x)=0\Rightarrow h(x)=c$$

Thus, a family of solutions is given by $\boxed{xe^{2y}-\sin xy+y^2+c=0}$.

3. Solve $(x+y)dx+x\ln xdy=0,\ \ x\in(0,\infty)$.

Solution:

$$M=x+y \qquad N=x\ln x$$

$$\frac{\delta M}{\delta y}=1 \qquad \frac{\delta N}{\delta x}=1+\ln x$$

$\Rightarrow$ not exact

Multiply the DE by $v(x,y)=\dfrac{1}{x}$:

$$\left(1+\frac{y}{x}\right)dx+\ln x dy=0$$

$$M=1+\frac{y}{x} \qquad N=\ln x$$

$$\frac{\delta M}{\delta y}=\frac{1}{x} \qquad \frac{\delta N}{\delta x}=\frac{1}{x}$$

$\Rightarrow$ exact

$$\frac{\delta F}{\delta x}=1+\frac{y}{x}$$

$$F(x,y)=x+y\ln x+g(y)$$

$$\frac{\delta F}{\delta y}=\ln x+g'(y)=\ln x \Rightarrow g'(y)=0 \Rightarrow g(y)=c$$

$\Rightarrow F(x,y)=x+y\ln x+c$ and $\boxed{x+y\ln x+c=0}$ is a solution of both DE.

2.5 The Linear Equation of Order One

If an equation is not exact we attempt to make it exact by introducing an ***integrating factor***.

The existence of an integrating factor can be demonstrated for the class of linear equations of order one. An equation that is linear and of order one in the dependent variable *y* must by definition (Section 1.2) be of the form

$$A(x)\frac{dy}{dx}+B(x)y=C(x) \qquad (1)$$

Dividing each side of (1) by $A(x)$, we get

$$\frac{dy}{dx}+P(x)y=Q(x) \qquad (2)$$

which we choose as the standard form for the linear equation of order one.

Suppose there exists for (2) a positive integrating factor $v(x)>0$, a function of *x* alone. Then

$$v(x)\left[\frac{dy}{dx}+P(x)y\right]=v(x)Q(x) \qquad (3)$$

must be an exact equation. But (3) can easily be put in the form

$$Mdx+Ndy=0$$

with

$$M=vPy-vQ \quad \text{and} \quad N=v,$$

in which *v¸ P,* and *Q* are functions of *x* alone.

Therefore if (3) is to be exact, it follows from the requirement

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

that *v* must satisfy the equation

$$vP = \frac{dv}{dx}. \qquad (4)$$

From (4), *v* may be obtained readily, for

$$Pdx = \frac{dv}{v},$$

so

$$\ln v = \int Pdx$$

or

$$v = e^{\int Pdx} \qquad (5)$$

This means that if (2) has a positive integrating factor independent of *y*, then the factor must be as given by (5). It remains to show that the *v* given by (5) is actually an integrating factor of

$$\frac{dy}{dx} + P(x)y = Q(x). \qquad (2)$$

Multiply (2) by the integrating factor to get

$$e^{\int Pdx}\frac{dy}{dx} + Pe^{\int Pdx}y = Qe^{\int Pdx}. \qquad (6)$$

The left side of (6) is the derivative of the product $ye^{\int Pdx}$; the right side of (6) is a function of *x* only. Hence (6) is exact.

Summary of Steps

(a) Put the equation into standard form $\frac{dy}{dx} + Py = Q$.
(b) Obtain the integrating factor $e^{\int Pdx}$.
(c) Multiply both sides of the equation in standard form by the integrating factor.
(d) Solve the resulting exact equation.

Note: In integrating the exact equation, *the integral of the left side is always the product of the dependent variable and the integrating factor used.*

Examples

1. Solve $x\dfrac{dy}{dx}-4y=x^6e^x$.

Solution:

Put the DE in standard form:

$$\frac{dy}{dx}-\frac{4}{x}y=x^5e^x$$

IF is $e^{-4\int\frac{dx}{x}}=e^{-4\ln|x|}=e^{\ln x^{-4}}=x^{-4}$.

Multiply by the IF:

$$x^{-4}\frac{dy}{dx}-4x^{-5}y=xe^x$$

$$d\left(x^{-4}y\right)=xe^x$$

$$\int d\left(x^{-4}y\right)dx=\int xe^x dx$$

$$x^{-4}y=xe^x-e^x+c$$

$$\boxed{y=x^5e^x-x^4e^x+cx^4}$$

2. (Exercise #18 p. 40) Solve $2y\left(y^2-x\right)dy=dx$

Solution:

Note: not linear in *y* but linear in *x*

Rewrite the DE:

$$2y^3-2yx=\frac{dx}{dy}$$

$$\frac{dx}{dy}+2yx=2y^3$$

IF is $e^{\int 2y dy}=e^{y^2}$.

Multiply by the IF:

$$e^{y^2}\frac{dx}{dy}+e^{y^2}2yx=e^{y^2}2y^3$$

$$d\left(e^{y^2}x\right)=2y^3e^{y^2}$$

$$\int d\left(e^{y^2}x\right)dy=\int 2y^3e^{y^2}dy$$

$$e^{y^2}x=\int 2y^3e^{y^2}dy$$

Integrate the right side by parts:

$$\begin{array}{lcl} y^2 & \longrightarrow & e^{y^2}2y \\ 2y & \searrow & e^{y^2} \end{array}$$

$$e^{y^2}x=y^2e^{y^2}-e^{y^2}+c$$

$$\boxed{x=y^2-1+ce^{-y^2}}$$

2.6 The General Solution of a Linear Equation

Consider the linear differential equation

$$\frac{dy}{dx}+P(x)y=Q(x). \qquad (1)$$

Suppose that P and Q are continuous functions on the interval $a<x<b$, and that $x=x_0$ is any number in that interval. If y_0 is an arbitrary real number, there exists a unique solution $y=y(x)$ of d.e. (1) that also satisfies the initial condition

$$y(x_0)=y_0.$$

Moreover, this solution satisfies (1) throughout the interval $a<x<b$.

Proof:

Multiply (1) by the integrating factor $v=e^{\int P dx}$ and integrate to get

$$yv=\int vQ\,dx\ +c.$$

Since $v\neq 0$ we can have

$$y=v^{-1}\int vQ\,dx\ +cv^{-1}. \qquad (2)$$

It is easy to show that since $v\neq 0$ and v is continuous on $a<x<b$, (2) is a family of solutions of (1). It is also easy to see that given any x_0 on the interval $a<x<b$ together with any number y_0, we can choose the constant c so that $y=y_0$ when $x=x_0$.

Exercises

Find the general solution.

2) $y'=x-2y$

Solution:

$$y'=x-2y$$

$$\frac{dy}{dx}+2y=x\Rightarrow \text{I.F. } e^{\int 2dx}=e^{2x}$$

$$e^{2x}\frac{dy}{dx}+2e^{2x}y=xe^{2x}$$

$$ye^{2x}=\int xe^{2x}dx$$

Using tabular integration by parts for the right side, we get

x	e^{2x}
1	$\frac{1}{2}e^{2x}$
	$\frac{1}{4}e^{2x}$

$$ye^{2x} = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$
$$y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}$$
$$4y = 2x - 1 + ce^{-2x}$$

4) $udx + (1-3u)\,xdu = 3u^2 e^{3u} du$

Solution:

$$udx + (1-3u)\,xdu = 3u^2 e^{3u} du$$
$$\frac{dx}{du} + \frac{1-3u}{u}x = 3ue^{3u}$$
$$\frac{dx}{du} + \left(\frac{1}{u} - 3\right)x = 3ue^{3u} \Rightarrow \text{I.F. } e^{\int\left(\frac{1}{u}-3\right)du} = e^{\ln u - 3u} = ue^{-3u}$$
$$ue^{-3u}dx + \left(\frac{1}{u} - 3\right)ue^{-3u}xdu = 3ue^{3u}ue^{-3u}du$$
$$d\left(xue^{-3u}\right) = 3u^2 du$$
$$xue^{-3u} = \int 3u^2 du$$
$$xue^{-3u} = u^3 + c$$
$$xu = \left(u^3 + c\right)e^{3u}$$