Math 275 Elementary Differential Equations

Chapter 1

1.1 Examples

Definitions

An equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more dependent variables, is called a **differential equation** (DE). A variable is called an **independent variable** when an equation contains one or more derivatives with respect to the variable; a variable is called a **dependent variable** if a derivative of this variable occurs.

Examples

1. $x\dfrac{dy}{dx} + y^2 = 0$

x – independent variable
y – dependent variable

The DE can also be written as

$$x + y^2\frac{dx}{dy} = 0$$

x – dependent variable
y – independent variable

2. $L\dfrac{di}{dt} + Ri = E$

i – dependent variable
t – independent variable
L, R, and E – **parameters**

3. $x\dfrac{\delta u}{\delta x} = -\dfrac{\delta v}{\delta y}$

x, y – independent variables
u, v – dependent variables

1.2 Definitions

There are ***three ways*** to classify DE:

By (1) type
(2) order
(3) linearity

Classification by Type

(a) ordinary differential equation (ODE)

- an equation containing only derivatives of one or more dependent variables, with respect to a single independent variable

Examples:

$$\frac{dy}{dx} - 4y = 1$$

$$\frac{dy}{dx} = \sin x$$
$$\frac{du}{dx} + \frac{dv}{dx} = x$$
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 6y = 0$$

(b) <u>partial differential equation</u> (PDE)

- an equation involving the derivatives of one or more dependent variables of two or more independent variables

Examples:

$$\frac{\delta u}{\delta x} = -\frac{\delta v}{\delta y}$$
$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = u$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\delta u}{\delta t}$$

Classification by Order

Definition The **<u>order</u>** of a differential equation is the order of the highest-ordered derivative appearing in the equation.

The equation

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0 \qquad (1)$$

is called an "*n*th order" ordinary differential equation.

We shall assume that it is always possible to solve for $y^{(n)}$ in terms of the other $n+1$ variables $x, y, y', \ldots, y^{(n-1)}$, i.e.

$$y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right). \qquad (2)$$

Examples

1. $\underbrace{\frac{d^2w}{dx^2}}_{2nd\text{ order}} - x\underbrace{\left(\frac{dw}{dx}\right)}_{1st\text{ order}}^4 - 4w = e^{2x} \Rightarrow$ DE is of order 2 or DE is a second-order equation

2. $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ is a second-order PDE

Classification by Linearity

Definition An ordinary differential equation of order n is called **linear** if it may be written in the form

$$b_0(x)\frac{d^n y}{dx^n}+b_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+b_{n-1}(x)\frac{dy}{dx}+b_n(x)y=R(x).$$

The notion of linearity may be extended to partial differential equations.

Note: Linear DE are characterized by two properties:
(i) The dependent variable y and all its derivatives are of the first degree, i.e. the power of each term involving y is 1.
(ii) Each coefficient depends only on the variable x.

Definition An equation that is not linear is said to be **nonlinear**.

Examples

$$xdy+ydx=0$$
$$y''-2y'+y=0$$
$$x^3\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+5y=e^x$$

are all linear DE, but

$yy''-2x\left(\frac{d^3y}{dx^3}\right)^2+y^3=0$ is nonlinear.

Solution of a DE

Definition A function ϕ defined on an interval $a<x<b$, is called a **solution** of the differential equation (2), provided that the n derivatives of the function exist on the interval $a<x<b$ and

$$\phi^{(n)}(x)=f\left(x,\phi(x),\phi'(x),\ldots,\phi^{(n-1)}(x)\right),$$

for every x in $a<x<b$.

Example 3 Show that $y=xe^x$ is a solution of the linear equation $y''-2y'+y=0$ on $(-\infty,\infty)$.

Solution:

$$\left.\begin{array}{l} y=xe^x \\ y'=xe^x+e^x \\ y''=xe^x+2e^x \end{array}\right\rangle \Rightarrow y''-2y'+y=xe^x+2e^x-2xe^x-2e^x+xe^x=0$$

Note:

1. The constant function $y=0$, $-\infty<x<\infty$, also satisfies the DE. This solution is called the **trivial solution**.
2. Not every DE necessarily has a solution.

Example 4 The first-order DE $\left(\frac{dy}{dx}\right)^2+1=0$ and $(y')^2+y^2+4=0$ do not have real solutions. Why?

Example 5 The second-order DE $(y'')^2+10y^4=0$ has only one real solution. What is it?

1.3 Families of Solutions

Definition A given DE usually possesses an infinite number of solutions, called a **family of solutions**.

In calculus, solutions of the first-order DE

$$\frac{dy}{dx}=f(x) \qquad (1)$$

are given by

$$y=\int f(x)\,dx+c\,, \qquad (2)$$

where c is an arbitrary constant. (2) is called a **one-parameter family** of solutions to (1), with c as the **parameter**.

Example 1 The DE

$$\frac{dy}{dx}=2\cos 3x$$

has the family of solutions

$$y=\frac{2}{3}\sin 3x+c\,.$$

Example 2 Show that the function $y=c_1\cos 4x+c_2\sin 4x$, where c_1 and c_2 are arbitrary constants, is a solution of the DE $y''+16y=0$.

Solution:

$$\left.\begin{aligned} y &= c_1\cos 4x+c_2\sin 4x\\ y' &= -4c_1\sin 4x+4c_2\cos 4x\\ y'' &= -16c_1\cos 4x-16c_2\sin 4x \end{aligned}\right\rangle \Rightarrow y''+16y=-16c_1\cos 4x-16c_2\sin 4x+16c_1\cos 4x+16c_2\sin 4x=0$$

$y=c_1\cos 4x+c_2\sin 4x$ is a **two-parameter** family of solutions to the DE $y''+16y=0$.

Note: When solving an nth-order DE $F\left(x,y,y',\ldots,y^{(n)}\right)=0$ we expect an **n-parameter family of solutions** $G(x,y,c_1,\ldots,c_n)=0$.

Definition A solution of a DE that is free of arbitrary parameters is called a **particular solution**.

One way of obtaining a particular solution is to choose specific values of the parameter(s).

Definition A DE along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an **initial-value problem** (IVP). The subsidiary conditions are **initial conditions**. If the subsidiary conditions are given at more than one value of the independent variable, the problem is a **boundary-value problem** (BVP) and the conditions are **boundary conditions**.

Example 3 $y'' + 2y' = e^x$; $y(\pi) = 1$, $y'(\pi) = 2$ is an IVP.

Example 4 $y'' + 2y' = e^x$; $y(0) = 1$, $y(1) = 1$ is a BVP.

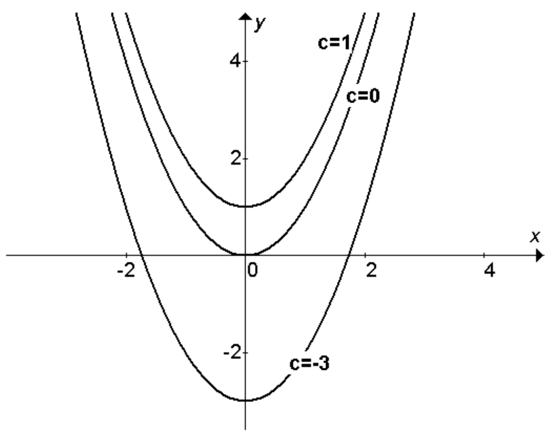
A solution to an IVP or BVP is a function that both solves the DE and satisfies all the given subsidiary conditions.

1.4 Geometric Interpretation

Solutions to a DE graph as a family of parallel curves.

Example 1 $\dfrac{dy}{dx} = 2x$ 1st-order DE

$y = x^2 + c$ solutions

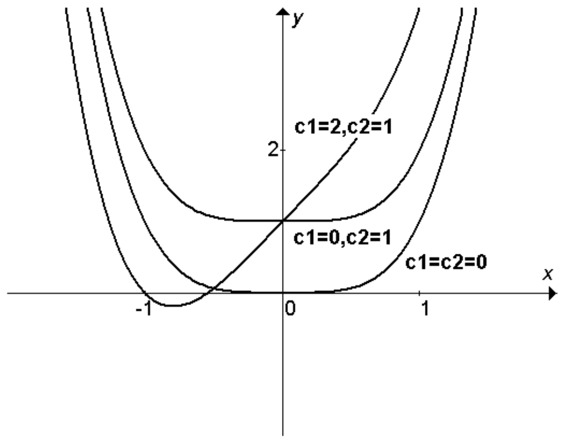


Note: Through each point in the plane there passes one and only one member of the family of solutions. We assume this is true for any first-order DE.

Example 2 (Ex 1.7 p 11)

$y'' = 12x^2$ 2nd-order DE

$y = x^4 + c_1 x + c_2$ family of solutions



Note: There are at least 2 solutions passing through $(0,1)$ and a point near $(-1/2,0)$. However these pairs of solutions do not have the same slope at their point of intersection. Thus, for second-order equations we have:

Through each point in the plane there passes one and only one member of the family of solutions that has a given slope.

1.5 The Isoclines of an Equation

Consider the first-order DE

$$\frac{dy}{dx} = f(x,y). \qquad (1)$$

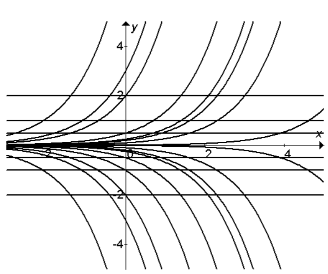
(1) assigns to each point $(a,b) \in D_f$ some direction or slope $f(a,b)$ and we can talk of the **direction field** or **slope field** of a DE.

A useful direction field can be constructed by evaluating *f* at each point of a rectangular grid consisting of at least a few hundred points. Then, at each point of the grid, a short line segment is drawn whose slope is the value of *f* at each point. This can be done systematically by first drawing curves called **isoclines**, i.e. curves along which the direction indicated by (1) is fixed.

Example 1 (Ex 1.8 p 12)

$$\frac{dy}{dx} = y$$

Isoclines are straight lines $f(x,y) = y = c$.



1.6 An Existence Theorem

Consider the initial value problem:

$$\text{Solve} \quad \frac{dy}{dx} = f(x,y) \quad (1)$$

s.t. $y(x_0) = y_0$.

Let T denote the rectangular region defined by

$$|x - x_0| \le a \quad \text{and} \quad |y - y_0| \le b,$$

a region with the point (x_0, y_0) at its center. Suppose that f and $\partial f/\partial y$ are continuous functions of x and y in T.

Under the conditions imposed on $f(x,y)$, an interval exists about x_0, $|x - x_0| \le h$, and a function $y(x)$ which has the properties:

(a) $y = y(x)$ is a solution of equation (1) on the interval $|x - x_0| \le h$.
(b) On the interval $|x - x_0| \le h$, $y(x)$ satisfies the inequality

$$|y(x) - y(x_0)| \le b.$$

(c) At $x = x_0$, $y = y(x_0) = y_0$.

(d) $y(x)$ is unique on the interval $|x - x_0| \le h$ in the sense that it is the only function that has all the properties (a), (b), and (c).

The interval $|x - x_0| \le h$ may or may not need to be smaller than the interval $|x - x_0| \le a$ over which conditions were imposed upon $f(x,y)$.

Roughly, the theorem says that if $f(x,y)$ is sufficiently well behaved near the point (x_0, y_0), then the differential equation (1) has a solution that passes through the point (x_0, y_0) and that solution is unique near (x_0, y_0).

Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x,y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0 $x_0 - h < x < x_0 + h,\ h > 0,$ contained in $a \le x \le b$, and a unique function $y(x)$, defined on I_0, that is a solution of the IVP

$$\frac{dy}{dx} = f(x,y)$$
$$\text{s.t.} \quad y(x_0) = y_0.$$