Math 275 Elementary Differential Equations

Chapter 14 The Laplace Transform

14.1 The Transform Concept

"transform" functions into other functions

Ex. $D: x^2 \to 2x$
$\int: \quad \sin x \to -\cos x$

Integral Transform

$$T\{F(t)\} = \int_{-\infty}^{\infty} K(s,t)F(t)\,dt = f(s)$$

$K(s,t)$ is called the **kernel** of the transformation.

If we choose

$$K(s,t) = \begin{cases} 0, & t < 0 \\ e^{-st}, & t \ge 0 \end{cases}$$

we have the **Laplace transform**.

14.2 Definition of the Laplace Transform

$$L\{F(t)\} = \int_0^{\infty} e^{-st}F(t)\,dt \qquad (1)$$

Or $$\int_0^{\infty} e^{-st}F(t)\,dt = f(s) \qquad (2)$$

14.3 Transforms of Elementary Functions

Ex1. $L\{e^{kt}\}$, where k is a constant

Solution:

$$L\{e^{kt}\} = \int_0^{\infty} e^{-st}e^{kt}\,dt$$
$$= \int_0^{\infty} e^{-(s-k)t}\,dt$$
$$= \left.\frac{-e^{-(s-k)t}}{s-k}\right]_0^{\infty}$$

(i) For $s \le k$, the exponent is + or 0 and the integral diverges

(ii) For $s > k$, the integral is equal to $\frac{1}{s-k}$

Thus,

$$\boxed{L\{e^{kt}\} = \frac{1}{s-k}, \quad s > k} \qquad (1)$$

For $k = 0$,

$$\boxed{L\{1\} = \frac{1}{s}, \quad s > 0} \qquad (2)$$

Ex2. $L\{\sin kt\}$, where k is a nonzero constant

Solution:

$$L\{\sin kt\} = \int_0^\infty e^{-st} \sin kt \; dt$$

$$= \left. \frac{e^{-st}\left(-s \sin kt - k \cos kt\right)}{s^2 + k^2} \right]_0^\infty \qquad (3)$$

For $s > 0$, $e^{-st} \to 0$ as $t \to \infty$. Also, $\sin kt$ and $\cos kt$ are bounded as $t \to \infty$. Thus (3) gives

$$L\{\sin kt\} = 0 - \frac{1(0-k)}{s^2 + k^2}$$

$$\boxed{L\{\sin kt\} = \frac{k}{s^2 + k^2}, \quad s > 0} \qquad (4)$$

Similarly,

$$\boxed{L\{\cos kt\} = \frac{s}{s^2 + k^2}, \quad s > 0} \qquad (5)$$

Ex3. $L\{t^n\}$, n is a positive integer

Solution:

$$L\{t^n\} = \int_0^\infty e^{-st} t^n dt$$

$$= \left. \frac{-t^n e^{-st}}{s} \right]_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt \qquad (6)$$

For $s > 0$ and $n > 0$, the 1st term is 0

$$\int_0^\infty e^{-st} t^n dt = \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt, \quad s > 0$$

$$L\{t^n\} = \frac{n}{s} L\{t^{n-1}\}, \quad s > 0 \qquad (7)$$

From (7), conclude that for $n \ge 1$

$$L\{t^{n-1}\} = \frac{n-1}{s} L\{t^{n-2}\}$$

so

$$L\left\{t^{n-1}\right\} = \frac{n(n-1)}{s^2} L\left\{t^{n-2}\right\} \qquad (8)$$

Iteration yields

$$L\left\{t^{n}\right\} = \frac{n(n-1)(n-2)\cdot\cdots\cdot 2\cdot 1}{s^n} L\left\{t^{0}\right\}$$
$$= \frac{n(n-1)(n-2)\cdot\cdots\cdot 2\cdot 1}{s^n}\cdot\frac{1}{s}$$

$$\boxed{L\left\{t^{n}\right\} = \frac{n!}{s^{n+1}}, \quad s>0} \qquad (9)$$

Note: L is a linear operator which means that

$$L\left\{c_1F_1(t)+c_2F_2(t)\right\} = c_1L\left\{F_1(t)\right\}+c_2L\left\{F_2(t)\right\}.$$

Ex4. Find $L\left\{4-3e^{2t}+5\sin 3t\right\}$.

Solution:

$$L\left\{4-3e^{2t}+5\sin 3t\right\}$$
$$= L\{4\}-3L\left\{e^{2t}\right\}+5L\{\sin 3t\}$$
$$= 4\cdot\frac{1}{s}-3\cdot\frac{1}{s-2}+5\cdot\frac{3}{s^2+9}$$
$$= \frac{4}{s}-\frac{3}{s-2}+\frac{15}{s^2+9}$$

Ex5. Find $L\left\{F(t)\right\}$, where

$$F(t)=\begin{cases} 2, & 0<t<5 \\ 0, & 5<t<10 \\ e^{4t} & 10<t \end{cases}$$

Solution:

$$f(s)=\int_0^\infty e^{-st}F(t)\,dt$$
$$=\int_0^5 2e^{-st}dt+\int_5^{10} e^{-st}\cdot 0\,dt+\int_{10}^\infty e^{-st}\cdot e^{4t}dt$$
$$=-\frac{2}{s}e^{-st}\Big]_0^5+\int_{10}^\infty e^{-(s-4)t}dt$$
$$=-\frac{2}{s}e^{-5s}+\frac{2}{5}+\lim_{b\to\infty}\left(-\frac{e^{-(s-4)b}}{s-4}\right)+\frac{e^{-10(s-4)}}{s-4}$$
$$=-\frac{2}{s}e^{-5s}+\frac{2}{5}+\frac{e^{-10(s-4)}}{s-4}$$

Note: $F(t)$ has jump discontinuities at $t=5$ and $t=10$, values which are reflected in the exponential forms e^{-5s} and $e^{-10(s-4)}$.

Existence of the Transform

14.4 Sectionally Continuous Functions

Def. A function $F(t)$ is said to be **sectionally continuous** over a closed interval $[a,b]$, if it is continuous at every point in $[a,b]$, except possibly for a finite number of points where $F(t)$ has a jump discontinuity.

14.5 Functions of Exponential Order

Def. A function $F(t)$ is said to be of **exponential order** b if there exist positive constants T and M such that

$$|F(t)| \le Me^{bt} \quad \text{for } t \ge T .$$

We also write

$$F(t) = O\left(e^{bt}\right) \text{ as } t \to \infty .$$

Note: The definition says that for some value of b, $F(t)$ grows no faster than a function of the form Me^{bt}.

Ex1. $F(t) = e^{5t}\sin 2t = O\left(e^{5t}\right)$ since $\left|e^{5t}\sin 2t\right| \le e^{5t}$. Take $M=1$ and T any positive constant.

Ex2. $F(t) = e^{t^2}$ is not of exponential order since

$$\lim_{t\to\infty}\frac{e^{t^2}}{e^{bt}} = \lim_{t\to\infty} e^{t(t-b)} = +\infty \text{ for any } b$$

$\Rightarrow e^{t^2}$ grows faster than e^{bt} for any choice of b

Conditions for the Existence of the Laplace Transform

Theorem If $F(t)$ is sectionally continuous on $(0,\infty)$ and of exponential order b then $L\{F(t)\}$ exists for $s > b$.

14.6 Functions of Class A

Functions of class A are functions that are

a) sectionally continuous on $[0,\infty)$
b) $O\left(e^{bt}\right)$ as $t \to \infty$

so a restatement of the theorem is

Theorem If $F(t)$ is of class *A*, then $L\{F(t)\}$ exists.

Properties of the Laplace Transform

1. $L\{e^{at}F(t)\} = f(s-a)$ **shift in *s***

2. Laplace transform of the derivative
   Let
   (i) $F(t)$ be continuous on $[0,\infty)$ and $O(e^{bt})$ as $t \to \infty$
   (ii) $F'(t)$ be class *A*'
   Then for $s > b$,
   $$L\{F'(t)\} = sL\{F(t)\} - F(0)\ .$$

   Using induction, we can extend this property to higher-order derivatives of $F(t)$.

   $$\begin{aligned} L\{F''(t)\} &= sL\{F'(t)\} - F'(0) \\ &= s\left[L\{F(t)\} - F(0)\right] - F'(0) \\ &= s^2 L\{F(t)\} - sF(0) - F'(0) \end{aligned}$$

   In general,

   $$L\{F^{(n)}(t)\} = s^n L\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - ... - F^{(n-1)}(0)$$

3. Derivatives of the Laplace transform
   If $F(t)$ is of class *A*. Then for $s > b$,
   $$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

Proofs:

1. $L\{e^{at}F(t)\} = \int_0^\infty e^{-st}e^{at}F(t)\,dt = \int_0^\infty e^{-(s-a)t}F(t)\,dt = f(s-a)$

2. $L\{F'(t)\} = \int_0^\infty e^{-st}F'(t)\,dt$
   Use integration by parts with

   $$u = e^{-st} \text{ and } dv = F'(t)\,dt$$
   $$\begin{aligned} &\Rightarrow \lim_{n\to\infty}\left[e^{-st}F(t)\right]_0^N + s\int_0^N e^{-st}F(t)\,dt \\ &= \lim_{n\to\infty} e^{-sN}F(N) - f(0) + s\lim_{n\to\infty}\int_0^N e^{-st}F(t)\,dt \\ &= \lim_{n\to\infty} e^{-sN}F(N) - f(0) + sL\{F(t)\} \end{aligned}$$

   Since $F = O(e^{bt})$ there exists a constant *M* such that for large *N*,

   $$\left|e^{-sN}F(N)\right| \le e^{-sN}Me^{bN} = Me^{-(s-b)N}$$

Hence, for $s > b$,

$$0 \le \lim_{n\to\infty}\left|e^{-sN}F(N)\right| \le \lim_{n\to\infty} Me^{-(s-b)N} = 0$$

Thus, $e^{-sN}F(N) \to 0$ as $N \to \infty$.

3.

Examples

1. $L\left\{e^{at}\sin bt\right\}$

Solution:

$$L\{\sin bt\} = \frac{b}{s^2 + b^2}$$

$$L\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2 + b^2}$$

2. Determine $L\{\cos bt\}$ using $L\{\sin bt\}$ and the transform of the derivative.

Solution:

$$F(t) = \sin bt$$

$$F'(t) = t\cos bt$$

$$L\{F'(t)\} = sL\{\sin bt\} - F(0)$$

$$L\{b\cos bt\} = \frac{sb}{s^2 + b^2}$$

$$bL\{\cos bt\} = \frac{sb}{s^2 + b^2}$$

$$L\{\cos bt\} = \frac{s}{s^2 + b^2}$$

3. Determine $L\{t\sin bt\}$.

Solution:

$$L\{\sin bt\} = \frac{b}{s^2 + b^2} = f(s)$$

$$f'(s) = \frac{-2bs}{\left(s^2 + b^2\right)^2}$$

$$\Rightarrow L\{t\sin bt\} = -f'(s) = \frac{2bs}{\left(s^2 + b^2\right)^2}$$

Inverse Laplace Transform

$$L : F(t) \to f(s)$$

$$L^{-1} : f(s) \to F(t)$$

Why do we need the inverse?

Example Consider the IVP

$$y'' - y = -t, \quad y(0) = 0, \quad y'(0) = 1$$

Take the Laplace transform of both sides

$$L\{y''\} - L\{y\} = L\{-t\}$$
$$L\{y''\} - L\{y\} = -\frac{1}{s^2}$$
$$L\{y''\} = s^2 L\{y\} - sy(0) - y'(0)$$
$$= s^2 Y(s) - 1, \text{ where } Y(s) = L\{y\}$$
$$s^2 Y(s) - 1 - Y(s) = -\frac{1}{s^2}$$

Solving for $Y(s)$,

$$Y(s) = \frac{1 - \frac{1}{s^2}}{s^2 - 1} = \frac{s^2 - 1}{s^2(s^2 - 1)} = \frac{1}{s^2} = L\{t\}$$
$\Rightarrow y(t) = t$ is a solution to the IVP

Definition Given a function $f(s)$, if there is a function $F(t)$ that is continuous on $[0, \infty)$ and satisfies

$$L\{F(t)\} = f(s)$$

then $F(t)$ is said to be the **inverse Laplace transform** of $f(s)$ and we write

$$F = L^{-1}\{f\}.$$

Ex1. Determine $L^{-1}\{f\}$.

a. $f(s) = \frac{2}{s^3}$

Solution:

$$f(s) = \frac{2!}{s^3} \Rightarrow L^{-1}\left\{\frac{2!}{s^3}\right\} = t^2$$

b. $f(s) = \frac{3}{s^2 + 9}$

Solution:

$$f(s) = \frac{3}{s^2 + 3^2} \Rightarrow L^{-1}\left\{\frac{3}{s^2 + 3^2}\right\} = \sin 3t$$

c. $f(s)=\dfrac{s-1}{s^2-2s+5}$

Solution:

$$f(s)=\frac{s-1}{(s-1)^2+2^2}\Rightarrow L^{-1}\{f\}=e^t\cos 2t$$

Linearity of the Inverse Transform

Theorem Assume $L^{-1}\{f\}$, $L^{-1}\{f_1\}$, and $L^{-1}\{f_2\}$ exist and are continuous on $[0,\infty)$ and let c be any constant. Then

$$L^{-1}\{f_1+f_2\}=L^{-1}\{f_1\}+L^{-1}\{f_2\}$$
$$L^{-1}\{cf\}=cL^{-1}\{f\}$$

Ex2. Determine $L^{-1}\left\{\dfrac{5}{s-6}-\dfrac{6s}{s^2+9}+\dfrac{3}{2s^2+8s+10}\right\}$.

Solution:

By linearity,

$$\begin{aligned}
&L^{-1}\left\{\frac{5}{s-6}-\frac{6s}{s^2+9}+\frac{3}{2s^2+8s+10}\right\}\\
&=5L^{-1}\left\{\frac{1}{s-6}\right\}-6L^{-1}\left\{\frac{s}{s^2+9}\right\}+\frac{3}{2}L^{-1}\left\{\frac{1}{s^2+4s+5}\right\}\\
&=5e^{6t}-6\cos 3t+\frac{3}{2}L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}\\
&=5e^{6t}-6\cos 3t+\frac{3}{2}e^{-2t}\sin t
\end{aligned}$$

Ex3. Determine $L^{-1}\left\{\dfrac{5}{(s+2)^4}\right\}$.

Solution:

The $(s+2)^4$ in the denominator suggests we work with the formula

$$L^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\}=e^{at}t^n$$

with $a=-2$ and $n=3$. Thus,

$$L^{-1}\left\{\frac{5}{(s+2)^4}\right\}=\frac{5}{6}L^{-1}\left\{\frac{3!}{(s+2)^4}\right\}=\frac{5}{6}e^{-2t}t^3.$$

Ex4. Determine $L^{-1}\left\{\dfrac{3s+2}{s^2+2s+10}\right\}$.

Solution:

The denominator is equal to $(s+1)^2+3^2$ so we can use one or both of

$$L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}=e^{at}\cos bt$$
$$L^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\}=e^{at}\sin bt.$$

Here, $a=-1,\ b=3$. Express

$$\frac{3s+2}{s^2+2s+10}=A\frac{s+1}{(s+1)^2+3^2}+B\frac{3}{(s+1)^2+3^2}$$

where A and B are constants to be determined.

$$3s+2=A(s+1)+3B$$
$$\Rightarrow\begin{cases}A=3\\A+3B=2\end{cases}\Rightarrow\begin{cases}A=3\\B=-1/3\end{cases}$$
$$L^{-1}\left\{\frac{3s+2}{s^2+2s+10}\right\}=3L^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\}-\frac{1}{3}L^{-1}\left\{\frac{3}{(s+1)^2+3^2}\right\}$$
$$=3e^{-t}\cos 3t-\frac{1}{3}e^{-t}\sin 3t$$

Ex5. Determine $L^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}$.

Solution:

$$\frac{7s-1}{(s+1)(s+2)(s-3)}=\frac{A}{s-1}+\frac{B}{s+2}+\frac{C}{s-3}$$
$$7s-1=A(s+2)(s-3)+B(s+1)(s-3)+C(s-1)(s+2)$$
$$s=-2:\quad -15=5B\Rightarrow B=-3$$
$$s=3:\quad 20=20C\Rightarrow C=1$$
$$s=-1:\quad -8=-4A\Rightarrow A=2$$

Thus,

$$L^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}$$
$$=2L^{-1}\left\{\frac{1}{s+1}\right\}-3L^{-1}\left\{\frac{1}{s+2}\right\}+L^{-1}\left\{\frac{1}{s-3}\right\}$$
$$=2e^{-t}-3e^{-2t}+e^{3t}$$

Ex6. Determine $L^{-1}\left\{\dfrac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

Solution:

$$\frac{s^2+9s+2}{(s-1)^2(s+3)}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+3}$$

$$s^2+9s+2=A(s-1)(s+3)+B(s+3)+C(s-1)^2$$

$s=1$: $\quad 12=4B\Rightarrow B=3$

$s=-3$: $\quad -16=16C\Rightarrow C=-1$

$s=0$: $\quad 2=-3A+3B+C$

$\quad 2=-3A+9-1\Rightarrow A=2$

Thus,

$$L^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$$
$$=2L^{-1}\left\{\frac{1}{s-1}\right\}+3L^{-1}\left\{\frac{1}{(s-1)^2}\right\}-L^{-1}\left\{\frac{1}{s+3}\right\}$$
$$=2e^t+3te^t-e^{-3t}$$

Ex7. Determine $L^{-1}\left\{\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

Solution:

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)}=\frac{A(s-1)+2B}{(s-1)^2+2^2}+\frac{C}{s+1}$$

$$2s^2+10s=\left[A(s-1)+2B\right](s+1)+C\left(s^2-2s+5\right)$$

$s=-1$: $\quad -8=8C\Rightarrow C=-1$

$s=1$: $\quad 12=4B+4C\Rightarrow 4B=16\Rightarrow B=4$

$s=0$: $\quad 0=-A+2B+5C\Rightarrow A=3$

Thus,

$$L^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$$
$$=3L^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\}+4L^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}-L^{-1}\left\{\frac{1}{s+1}\right\}$$
$$=3e^t\cos 2t+4e^t\sin 2t-e^{-t}$$

Solving Initial Value Problems

Previous Methods:

First find the general solution and then use the initial conditions to get the desired solution.

Laplace Transform Method:

There is no need to find the general solution.

Other Advantages:

- Can handle equations with jump discontinuities
- Can be used for linear DE with variable coefficients
- Can be used for systems of DE
- Can be used for PDE

Steps:

1. Take the Laplace transform of both sides of the equation.
2. Use the properties of the Laplace transform and the initial conditions to obtain an equation for the LT of the solution and then solve this equation for the transform.
3. Determine the inverse LT.

Ex1. Solve: $y'' - 2y' + 5y = -8e^{-t}$; $y(0) = 2$, $y'(0) = 12$

Solution:

$$L\{y'' - 2y' + 5y\} = L\{-8e^{-t}\}$$

$$L\{y''\} - 2L\{y'\} + 5L\{y\} = -8L\{e^{-t}\}$$

$$s^2L\{y\} - sy(0) - y'(0) - 2sL\{y\} + 2y(0) + 5L\{y\} = \frac{-8}{s+1}$$

$$\left(s^2 - 2s + 5\right)L\{y\} = 2s + 8 - \frac{8}{s+1}$$

$$\left(s^2 - 2s + 5\right)L\{y\} = \frac{2s^2 + 10s}{s+1}$$

$$L\{y\} = \frac{2s^2 + 10s}{\left(s^2 - 2s + 5\right)(s+1)}$$

$$y = L^{-1}\left\{\frac{2s^2 + 10s}{\left(s^2 - 2s + 5\right)(s+1)}\right\}$$

$$y = 3e^t \cos 2t + 4e^t \sin t - e^{-t} \quad \text{by Ex7}$$

Ex2. Solve: $y'' + 4y' - 5y = te^t$; $y(0) = 1$, $y'(0) = 0$

Solution:

$$L\{y''\} + 4L\{y'\} - 5L\{y\} = L\{te^t\}$$

$$s^2L\{y\} - sy(0) - y'(0) + 4sL\{y\} - 4y(0) - 5L\{y\} = \frac{1}{(s-1)^2}$$

$$\left(s^2 - 4s - 5\right)L\{y\} = s + 4 + \frac{1}{(s-1)^2}$$

$$L\{y\} = \frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3}$$

$$\frac{s^3+2s^2-7s+5}{(s+5)(s-1)^3}=\frac{A}{s+5}+\frac{B}{s-1}+\frac{C}{(s-1)^2}+\frac{D}{(s-1)^3}$$

$$\Rightarrow A=\frac{35}{216},\ \ B=\frac{181}{216},\ \ C=-\frac{1}{36},\ \ D=\frac{1}{6}$$

$$Y(s)=\frac{35}{216}\left(\frac{1}{s+5}\right)+\frac{181}{216}\left(\frac{1}{s-1}\right)-\frac{1}{36}\frac{1}{(s-1)^2}+\frac{1}{12}\frac{1}{(s-1)^3}$$

$$y(t)=\frac{35}{216}e^{-5t}+\frac{181}{216}e^{t}-\frac{1}{36}te^{t}+\frac{1}{12}t^2e^{t}$$

Ex3. Solve $w''(t)-2w'(t)+5w(t)=-8e^{\pi-t};\ \ w(\pi)=2,\ \ w'(\pi)=12$

Solution:

Note that the initial conditions are not at $t=0$ so move them to $t=0$.

Set $y(t)=w(t+\pi)$

$y'(t)=w'(t+\pi)$

$y''(t)=w''(t+\pi)$

Replace t by $t+\pi$ in the DE.

$$w''(t+\pi)-2w'(t+\pi)+5w(t+\pi)=-8e^{-t}$$

$$y''(t)-2y'(t)+5y(t)=-8e^{-t};\ \ y(0)=2,\ \ y'(0)=12$$

Ex 1 $\Rightarrow y(t)=3e^{t}\cos 2t+4e^{t}\sin 2t-e^{-t}$

Since $w(t+\pi)=y(t)$ then $w(t)=y(t-\pi)$.

Thus, $t\to t-\pi$:

$$w(t)=y(t-\pi)=3e^{t-\pi}\cos\left[2(t-\pi)\right]+4e^{t-\pi}\sin\left[2(t-\pi)\right]-e^{-(t-\pi)}$$

$$w(t)=3e^{t-\pi}\cos 2t+4e^{t-\pi}\sin 2t-e^{-(t-\pi)}$$

Transforms of Discontinuous Functions

Def. The **unit step function** $\alpha(t)$ is defined by

$$\alpha(t)=\begin{cases}0, & t<0\\ 1, & t\ge 0\end{cases}$$

The jump can be moved to another location by shifting the argument, i.e.,

$$\alpha(t-c)=\begin{cases}0, & t<c\\ 1, & t\ge c\end{cases}$$

The height of the jump can be modified by multiplying by a constant M

$$M\alpha(t-c)=\begin{cases}0, & t<c\\ M, & t\ge c\end{cases}$$

Any piecewise continuous function can be expressed in terms of unit step functions.

Ex1. $y(t)=\begin{cases} t^2, & 0<t<2 \\ 6, & t>2 \end{cases}$

$$\begin{aligned} y(t) &= t^2 - t^2\alpha(t-2)+6\alpha(t-2) \\ &= t^2 + \left(6-t^2\right)\alpha(t-2) \end{aligned}$$

Ex2. $f(t)=\begin{cases} 3, & 0<t<2 \\ 1, & 2<t<5 \\ t, & 5<t<8 \\ \dfrac{t^2}{10}, & t>8 \end{cases}$

$$f(t)=3-2\alpha(t-2)+(t-1)\alpha(t-5)+\left(\frac{t^2}{10}-t\right)\alpha(t-8)$$

Remark For $c\ge 0$, $L\{\alpha(t-c)\}=\dfrac{e^{-cs}}{s}$, since for $s>0$

$$\begin{aligned} L\{\alpha(t-c)\} &= \int_0^\infty e^{-st}\alpha(t-c)\,dt \\ &= \int_c^\infty e^{-st}dt = \lim_{N\to\infty} \left.\frac{-e^{-st}}{s}\right]_e^N \\ &= \frac{e^{-cs}}{s} \end{aligned}$$

Conversely,

$$L^{-1}\left\{\frac{e^{-cs}}{s}\right\}=\alpha(t-c),\ c>0$$

Theorem (Translation in *t*)

Let $f(s)=L\{F(t)\}$ exist for $s>a\ge 0$. If c is a positive constant, then

$$L\{F(t-c)\alpha(t-c)\}=e^{-cs}f(s), \qquad (1)$$

and conversely,

$$L^{-1}\left\{e^{-cs}f(s)\right\}=F(t-c)\alpha(t-c) \qquad (2)$$

Proof:

$$\begin{aligned} L\{F(t-c)\alpha(t-c)\} &= \int_0^\infty e^{-st}F(t-c)\alpha(t-c)\,dt \\ &= \int_c^\infty e^{-st}F(t-c)\,dt; \quad \text{let } v=t-c,\ dv=dt \\ &= \int_0^\infty e^{-as}e^{-sv}F(v)\,dv \\ &= e^{-as}\int_0^\infty e^{-sv}F(v)\,dv = e^{-as}f(s) \end{aligned}$$

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In practice it is more common to be faced with the problem of computing $L\{G(t)\alpha(t-c)\}$ rather than $L\{F(t-c)\alpha(t-c)\}$. Thus, (1) becomes

$$L\{G(t)\alpha(t-c)\} = e^{-cs}L\{G(t+c)\} \qquad (3)$$

if we identify $G(t)=F(t-c),\ \ G(t+c)=F(t)$.

Ex3. Determine $L\{t^2\alpha(t-1)\}$.

Solution:

To apply (3), take $G(t)=t^2$ and $c=1$. Then

$$G(t+c)=G(t+1)=(t+1)^2=t^2+2t+1$$
$$L\{G(t+c)\}=L\{t^2+2t+1\}=\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}$$

Thus,

$$L\{t^2\alpha(t-1)\}=e^{-s}\left[\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right].$$

Ex4. Example 15.10 p.288

Determine $L\{y(t)\}$, where

$$y(t)=\begin{cases} t^2, & 0<t<2 \\ 6, & t>2 \end{cases}$$

Solution:

$$y(t)=t^2-t^2\alpha(t-2)+6\alpha(t-2)$$
$$y(t)=t^2+\underbrace{(6-t^2)\alpha(t-2)}_{*}$$
$$*G(t)=6-t^2,\ \ c=2$$
$$G(t+c)=6-(t+2)^2=6-t^2-4t-4$$
$$=2-4t-t^2$$
$$L\{G(t+2)\}=\frac{2}{s}-\frac{4}{s^2}-\frac{2}{s^3}$$
$$L\{*\}=\frac{2}{s}e^{-2s}-\frac{4}{s^2}e^{-2s}-\frac{2}{s^3}e^{-2s}$$
$$L\{y(t)\}=L\{t^2\}+L\{*\}$$
$$=\frac{2}{s^3}-\frac{2}{s^3}e^{-2s}-\frac{4}{s^2}e^{-2s}-\frac{2}{s^3}e^{-2s}$$

Ex5 Determine $L^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$ and sketch its graph.

Solution:

In (2), $e^{-cs} = e^{-2s} \Rightarrow c = 2$

$$f(s) = \frac{1}{s^2} \Rightarrow F(t) = t .$$

Thus by (2)

$$L^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = F(t-2)\alpha(t-2) = (t-2)\alpha(t-2) .$$

Ex6 Solve the problem

$I''(t) + 4I(t) = g(t);\ \ I(0) = 0,\ I'(0) = 0$, where

$$g(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

Solution:

Express $g(t)$ as

$$g(t) = \alpha(t) - 2\alpha(t-1) + \alpha(t-2) .$$

Take the LT of both sides of the DE:

$$L\{I''(t)\} + 4L\{I'(t)\} = L\{g(t)\}$$

$$s^2 L\{I(t)\} + 4L\{I(t)\} = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$

$$L\{I(t)\} = \frac{1}{s(s^2+4)} - \frac{2e^{-s}}{s(s^2+4)} + \frac{e^{-2s}}{s(s^2+4)}$$

$$= f(s) - 2e^{-s} f(s) + e^{-2s} f(s)$$

where

$$f(s) = \frac{1}{s(s^2+4)} = \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{s}{s^2+4}$$

$$\Rightarrow \qquad F(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

Thus,

$$I(t) = L^{-1}\{f(s) - 2e^{-s} f(s) + e^{-2s} f(s)\}$$

$$= F(t) - 2F(t-1)\alpha(t-1) + F(t-2)\alpha(t-2)$$

$$= \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right) - \left[\frac{1}{2} - \frac{1}{2}\cos 2(t-1)\right]\alpha(t-1) + \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-2)\right]\alpha(t-2)$$

Convolution

Consider the IVP

$$y'' + y = g(t);\ \ y(0) = 0,\ y'(0) = 0\,.$$

Let $L\{y\} = Y(s)$ and $L\{g\} = G(s)$. Then taking the LT of both sides of the DE yields

$$s^2 Y(s) + Y(s) = G(s)$$

$$\text{or} \qquad Y(s) = \frac{1}{s^2+1} G(s)$$

$$= L\{\sin t\} \cdot L\{g\}$$

To find a formula for $y(t)$ in terms of $\sin t$ and $g(t)$.

Def. Let $F(t)$ and $G(t)$ be piecewise continuous on $[0,\infty)$. The **convolution** of $F(t)$ and $G(t)$, denoted by $F*G$, is defined by

$$(F*G)(t) = \int_0^t F(t-\beta) G(\beta)\, d\beta\,.$$

Ex0. $F(t) = t,\ G(t) = t^2$

$$(F*G)(t) = \int_0^t (t-\beta)\beta^2 d\beta$$

$$= \int_0^t \left(t\beta^2 - \beta^3\right) d\beta = \frac{t\beta^3}{3} - \frac{\beta^4}{4}\Bigg]_0^t$$

$$= \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}$$

Theorem (Convolution Theorem)

If $F(t)$ and $G(t)$ are functions of class A and $L\{F(t)\} = f(s)$ and $L\{G(t)\} = g(s)$, then

$$L\{F*G\} = f(s) g(s)$$

or equivalently,

$$L^{-1}\{f(s) g(s)\} = F*G\,.$$

Proof:

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt \qquad (1)$$

$$g(s) = \int_0^\infty e^{-st} G(\beta)\, d\beta \qquad (2)$$

$$f(s) g(s) = \int_0^\infty e^{-s\beta} f(s) G(\beta) d\beta \qquad (3)$$

$$L^{-1}\{f(s)\} = F(t) \Rightarrow L^{-1}\{e^{-s\beta} f(s)\} = F(t-\beta)\alpha(t-\beta)$$

$$\Rightarrow e^{-s\beta} f(s) = \int_0^\infty e^{-st} F(t-\beta)\alpha(t-\beta)\, dt$$

and (3) can be put in the form

$$f(s)g(s)=\int_0^\infty\int_0^\infty e^{-st}G(\beta)F(t-\beta)\alpha(t-\beta)\,dt\ d\beta$$
$$=\int_0^\infty\int_\beta^\infty e^{-st}G(\beta)F(t-\beta)\,dt\ d\beta$$

Since F and G are functions of class A, then it is ok to interchange the order of integration (proof in advanced calculus) to

$$f(s)g(s)=\int_0^\infty\int_0^t e^{-st}G(\beta)F(t-\beta)\,d\beta\,dt$$
$$=\int_0^\infty e^{-st}\left[\int_0^t G(\beta)F(t-\beta)\,d\beta\right]dt=L\left\{\int_0^t G(\beta)F(t-\beta)\,d\beta\right\}$$

Back to the IVP,

$$Y(s)=\frac{1}{s^2+1}G(s)$$
$$=L\{\sin t\}\cdot L\{g(t)\}$$
$$y(t)=\sin t*g(t)$$
$$=\int_0^t \sin(t-\beta)g(\beta)\,d\beta$$

Ex1. Solve the IVP

$$y''-y=g(t);\quad y(0)=1,\ y'(0)=1.$$

Solution:

$$s^2Y(s)-s-1-Y(s)=G(s)$$
$$Y(s)=\frac{s+1}{s^2-1}+\frac{1}{s^2-1}G(s)$$
$$=\frac{1}{s-1}+\frac{1}{s^2-1}G(s)$$
$$y(t)=e^t+L^{-1}\left\{\frac{1}{s^2-1}G(s)\right\}$$
$$=e^t+\sinh t*g(t)$$
$$y(t)=e^t+\int_0^t \sinh(t-\beta)g(\beta)\,d\beta$$

Ex2. Use the CT to find $L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$.

Solution:

$$L^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right\}$$
$$= \sin t * \sin t$$
$$= \int_0^t \sin(t-\beta)\sin\beta d\beta$$
$$= \frac{1}{2}\int_0^t \left[\cos(2\beta - t) - \cos t\right]^* d\beta$$
$$= \frac{1}{2}\left[\frac{\sin(2\beta - t)}{2}\right]_0^t - \frac{1}{2}t\cos t$$
$$= \frac{1}{2}\left[\frac{\sin t}{2} - \frac{\sin(-t)}{2}\right] - \frac{1}{2}t\cos t$$
$$= \frac{\sin t - t\cos t}{2}$$

$*$Use the formula $\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\beta-\alpha) - \cos(\beta+\alpha)\right]$

Solving Linear Systems with Laplace Transforms

Systems of linear DEs reduced to systems of linear algebraic equations

Ex1. Solve the IVP

$$\begin{cases} x'(t) - 2y(t) = 4t \\ y'(t) + 2y(t) - 4x(t) = -4t - 2 \end{cases} \qquad (1)$$
$$x(0) = 4, \quad y(0) = -5$$

Solution:

$$\begin{cases} L\{x'(t)\} - 2L\{y(t)\} = \dfrac{4}{s^2} \\ L\{y'(t)\} + 2L\{y(t)\} - 4L\{x(t)\} = -\dfrac{4}{s^2} - \dfrac{2}{s} \end{cases} \qquad (2)$$

Let $X(s) = L\{x(t)\}$ and $Y(s) = L\{y(t)\}$. Then

$$L\{x'(t)\} = sX(s) - x(0) = sX(s) - 4$$
$$L\{y'(t)\} = sY(s) - y(0) = sY(s) + 5$$

Substitute these in (2):

$$\begin{cases} sX(s) - 2Y(s) = \dfrac{4s^2+4}{s^2} \\ -4X(s) + (s+2)Y(s) = -\dfrac{5s^2+2s+4}{s^2} \end{cases} \qquad (3)$$

Then

$$X(s) = \frac{4s-2}{(s+4)(s-2)} = \frac{3}{s+4} + \frac{1}{s-2}$$

$$\Rightarrow x(t) = 3e^{-4t} + e^{2t}$$

From (1)

$$y(t) = \frac{1}{2}x'(t) - 2t$$

$$y(t) = -6e^{-4t} + e^{2t} - 2t$$