Math 275 Elementary Differential Equations

Chapter 11 Linear Systems of Equations

11.2 First-Order Systems with Constant Coefficients

We will show how matrix algebra can be used to solve systems of DE.

First note that a system of linear equations of order higher than the first can be written in terms of a first-order system.

<u>Example</u>

Consider the equation $y'' + 2y' - y = e^x$. If we let $u = y'$, then the equation becomes $u' = y - 2u + e^x$. Thus, the single second-order equation was replaced by the first-order system

$$\begin{aligned} y' &= u \\ u' &= y - 2u + e^x. \end{aligned}$$

Exercises

Replace the given DE by a system of first-order equations.

2) $y'' + 4y' + 4y = e^x$

Solution:

Let $u = y'$. Then the DE becomes

$$u' = -4u - 4y + e^x$$

and is replaced by the system

$$\begin{aligned} y' &= u \\ u' &= -4u - 4y + e^x \end{aligned}$$

4) $y''' + py'' + qy' + ry = f(x)$

Solution:

Let $u = y'$
$v = u' = y''$

Then the DE becomes

$$v' = -pv - qu - ry + f(x)$$

and is replaced by the system

$$\begin{aligned} y' &= u \\ u' &= v \\ v' &= -pv - qu - ry + f(x) \end{aligned}$$

Replace the given system by an equivalent system of first-order equations.

6) $v' - 2v + 2w' = 2 - 4e^{2x}$; $2v' - 3v + 3w' - w = 0$

Solution:

$$v' - 2v + 2w' = 2 - 4e^{2x} \qquad (1)$$
$$2v' - 3v + 3w' - w = 0 \qquad (2)$$
$$-2(1) + (2):$$
$$-2v' + 4v - 4w' = -4 + 8e^{2x}$$
$$\underline{2v' - 3v + 3w' - w = 0}$$
$$v - w' - w = -4 + 8e^{2x}$$
$$\boxed{w' = v - w + 4 - 8e^{2x}} \qquad (3)$$
$$-3(1) + 2(2):$$
$$-3v' + 6v - 6w' = -6 + 12e^{2x}$$
$$\underline{4v' - 6v + 6w' - 2w = 0}$$
$$v' - 2w = -6 + 12e^{2x}$$
$$\boxed{v' = 2w - 6 + 12e^{2x}} \qquad (4)$$

8) $(D^2 + 6)y + Dv = 0$; $(D + 2)y + (D - 2)v = 2$

Solution:

$$(D^2 + 6)y + Dv = 0 \longrightarrow y'' + 6y + v' = 0$$
$$(D + 2)y + (D - 2)v = 2 \longrightarrow y' + 2y + v' - 2v = 2$$

Let $u = y'$

$$u' + 6y + v' = 0$$
$$u + 2y + v' - 2v = 2 \Rightarrow \boxed{v' = 2v - u - 2y + 2}$$
$$u' = -6y - 2v + u + 2y - 2$$
$$\boxed{u' = u - 2v - 4y - 2}$$

11.3 Solution of a First-Order System

Consider the first-order system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -2x + 3y. \end{aligned} \qquad (1)$$

We can rewrite this system in the form

$$\begin{aligned} Dx - y &= 0, \\ 2x + (D - 3)y &= 0. \end{aligned} \qquad (2)$$

Apply the operator $D - 3$ to the first and add the two equations to eliminate y:

$$\left(D^2 - 3D + 2\right)x = 0. \qquad (3)$$

Similarly, we can eliminate x and get

$$\left(D^2 - 3D + 2\right)y = 0. \qquad (4)$$

Thus, the solutions of (1) are of the form

$$x = c_1 e^{2t} + c_2 e^t,$$
$$y = c_3 e^{2t} + c_4 e^t,$$

where there are some relations among the four constants c_1, c_2, c_3, c_4 that we can determine by substituting back into (1).

Alternative Way
Expect from the beginning the existence of solutions of the form

$$\begin{aligned} x &= c_1 e^{mt}, \\ y &= c_2 e^{mt}, \end{aligned} \qquad (5)$$

where the constants $c_1, c_2,$ and m must be determined by substitution into (1). Doing this we find

$$c_1 m e^{mt} = c_2 e^{mt}$$

and

$$c_2 m e^{mt} = -2c_1 e^{mt} + 3c_2 e^{mt},$$

or

$$\begin{aligned} -mc_1 + c_2 &= 0, \\ -2c_1 + (3 - m)c_2 &= 0. \end{aligned} \qquad (6)$$

The system (6) can have nontrivial solutions for c_1 and c_2 only if

$$\begin{vmatrix} -m & 1 \\ -2 & 3 - m \end{vmatrix} = 0 \qquad (7)$$

i.e.

$$m^2 - 3m + 2 = (m - 1)(m - 2) = 0.$$

Moreover, for the choice $m = 1$, (6) yields the condition $c_2 = c_1$, and for $m = 2$ we have $c_2 = 2c_1$. Thus there would be two distinct solutions of the form of (5), namely

$$\begin{aligned} x &= c_1 e^t, \\ y &= c_1 e^t \end{aligned} \qquad (8)$$

and

$$x = c_1 e^{2t}, \quad y = 2c_1 e^{2t}. \tag{9}$$

11.4 Some Matrix Algebra

A **matrix** is a rectangular array of numbers. Each of the numbers in the matrix is called an **element** of the matrix. A matrix is said to be of **dimension** $n \times m$ if has n rows and m columns. The general $n \times m$ matrix can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

where a_{ij} indicates the number in the ith row and jth column.

Two matrices of the same dimension are **equal** if the corresponding elements are equal.

Matrix addition is defined only for matrices of the same dimension; simply add corresponding elements.

Multiplication by a scalar can be performed on any matrix; simply multiply each entry by the scalar.

The **product** of a $1 \times n$ row vector and an $n \times 1$ column vector is given by the familiar **scalar product**

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n . \tag{4}$$

The **product** of two matrices is defined in terms of the scalar products of the row and column vectors. An $n \times m$ matrix and a $p \times q$ matrix can be multiplied only if $m = p$, i.e. the number of columns in the first matrix must equal the number of rows in the second matrix. The resulting product matrix has dimension $n \times q$.

$$A \cdot B = \begin{pmatrix} A_1 \cdot B^1 & A_1 \cdot B^2 & \dots & A_1 \cdot B^n \\ A_2 \cdot B^1 & A_2 \cdot B^2 & \dots & A_2 \cdot B^n \\ & & \vdots & \\ A_n \cdot B^1 & A_n \cdot B^2 & \dots & A_n \cdot B^n \end{pmatrix} \tag{5}$$

where A_i is the i th row vector of the matrix A

and B^j is the j th column vector of the matrix B.

Notation

Consider the system

$$\frac{dx}{dt} = 2x + y,$$
$$\frac{dy}{dt} = x - y.$$

This system can be written as

$$X' = AX$$

where $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $X' = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix}$.

Capital letters will be used for matrices and lowercase letters will be used for numbers.

A large zero will be used for the **zero matrix**, a matrix whose elements are all 0.
The letter *I* will be used for the **identity matrix**, a **square** matrix whose elements are 1's on the main diagonal and 0's everywhere else.

Some properties of the *algebra of matrices*:

$$(A+B)+C = A+(B+C)$$
$$A+B = B+A$$
$$A+O = A$$
$$A+(-1)A = O$$
$$(AB)C = A(BC)$$
$$A(B+C) = AB+AC$$
$$IA = AI = A$$
$$k(A+B) = kA+kB$$
$$kO = O$$
$$k(AB) = (kA)B = A(kB)$$

Theorem 11.1 Let *A* be an $n \times n$ matrix of constant real numbers and let *X* be an *n*-dimensional column vector. The system of equations

$$AX = O$$

has nontrivial solutions, i.e., $X \neq O$, if and only if the determinant of *A* is zero.

11.5 First-Order Systems Revisited

Let *X* be an *n*-dimensional column vector function of *t* and let *A* be an $n \times n$ matrix of real numbers. Suppose $B(t)$ is a column vector whose components are known functions of *t*. Then the vector equation

$$X' = AX + B \qquad (1)$$

represents a system of n equations in the n unknown component functions of X. If $B = O$, we say that the system is **homogeneous**.

From our past experience we have reason to believe that the homogeneous system

$$X' = AX \tag{2}$$

may have solutions of the form

$$X = Ce^{mt}, \tag{3}$$

where C is a constant vector and m is some number that we wish to determine. Substitution of X into (2) yields

$$Cme^{mt} = ACe^{mt},$$

which can be written

$$(AC - mC)e^{mt} = 0,$$

or, remembering that $C = IC$, in the form

$$(A - mI)Ce^{mt} = O. \tag{4}$$

(4) is to be satisfied for all real values of t, a condition that can be satisfied only if

$$(A - mI)C = O. \tag{5}$$

Now (5) will have nontrivial solutions only if the determinant of $A - mI$ is zero, i.e.

$$|A - mI| = 0. \tag{6}$$

Equation (6) is a polynomial equation of degree n in the unknown number m. Note that the polynomial $|A - mI|$ depends only on the matrix A. The polynomial $|A - mI|$ is called the **characteristic polynomial** of the matrix A and equation (6) is called the **characteristic equation** of the matrix A.

The roots of the characteristic equation of A are called the **eigenvalues** of A. A nonzero vector C_1, which is a solution of (5) for a particular eigenvalue m_1, is called an **eigenvector** of the matrix A corresponding to the eigenvalue m_1.

Back to the example in 11.3:

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -2x + 3y. \end{aligned} \tag{7}$$

In matrix notation,

$$X' = AX$$

where $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.

The characteristic equation of A is

$$\left| \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} - m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} -m & 1 \\ -2 & 3-m \end{vmatrix} = m^2 - 3m + 2 = 0$$

$\Rightarrow$ eigenvalues are $m_1 = 1$ and $m_2 = 2$ (real and distinct)

For $m_1 = 1$, (5) becomes

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = O \Rightarrow -c_1 + c_2 = 0 \Rightarrow c_1 = c_2$$

$$\Rightarrow C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ eigenvectors are scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $m_1 = 2$, (5) becomes

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = O \Rightarrow -2c_1 + c_2 = 0 \Rightarrow c_2 = 2c_1$$

$$\Rightarrow C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\Rightarrow$ eigenvectors are scalar multiples of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Thus, we get two distinct sets of solutions for (7):

$$X_1 = b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} \quad \text{and} \quad X_2 = b_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

where b_1 and b_2 are arbitrary constants. It is simple to verify that these are solutions of (7).

Claim Every solution of (7) is a linear combination of X_1 and X_2.

A set of m constant vectors of dimension n,

$$\{X_1, X_2, \ldots, X_m\},$$

is **linearly independent** if

$$c_1 X_1 + c_2 X_2 + \cdots + c_m X_m = O$$

implies that $c_1 = c_2 = \cdots = c_m = 0$.

A set of vector functions of *t*,

$$\{X_1(t), X_2(t), \ldots, X_m(t)\},$$

is **<u>linearly independent</u>** on an interval $a < t < b$ if

$$c_1 X_1(t) + c_2 X_2(t) + \cdots + c_m X_m(t) = O$$

for all *t* on the interval implies that

$$c_1 = c_2 = \cdots = c_m = 0 .$$

Theorem 11.2 If $X_1(t), \ldots, X_m(t)$ are each solutions of a homogeneous linear system $X' = AX$, then $c_1 X_1(t) + c_2 X_2(t) + \cdots + c_m X_m(t)$ is a solution of the same system for arbitrary constants $c_1, \cdots, c_m$.

Theorem 11.3 If *A* is an $n \times n$ matrix of real numbers and $\{X_1, \ldots, X_n\}$ is a linearly independent set of solutions of the system $X' = AX$ on the interval $a < t < b$, then any solution of the system is a unique linear combination of the set $\{X_1, \ldots, X_n\}$.

If the set of *n*-dimensional column vectors $\{X_1(t), \ldots, X_n(t)\}$ is considered as an $n \times n$ matrix

$$\begin{pmatrix} X_1(t) & X_2(t) & \ldots & X_n(t) \end{pmatrix},$$

then the determinant

$$\begin{vmatrix} X_1(t) & X_2(t) & \ldots & X_n(t) \end{vmatrix}$$

is called the Wronskian of the set of vectors.

Theorem 11.4 The set of *n*-dimensional column vectors $\{X_1(t), \ldots, X_n(t)\}$ is linearly independent at $t = t_0$ if, and only if, the Wronskian of the set is not zero at $t = t_0$, i.e.

$$W\{X_1(t_0), X_2(t_0), \ldots, X_n(t_0)\} = \begin{vmatrix} X_1(t_0) & X_2(t_0) & \ldots & X_n(t_0) \end{vmatrix} \neq 0.$$

Theorem 11.5 If the vector functions $X_1(t), \ldots, X_n(t)$ are solutions of the system $X' = AX$ for all *t* on the interval $a < t < b$ where *A* is an $n \times n$ matrix, then the set $\{X_1(t), \ldots, X_n(t)\}$ is linearly independent on $a < t < b$ if, and only if, $W\{X_1(t_0), X_2(t_0), \ldots, X_n(t_0)\} \neq 0$ for some t_0 on the interval $a < t < b$.

Theorem 11.6 If $X_1(t), \ldots, X_n(t)$ are linearly independent solutions of the *n*-dimensional homogeneous system $X' = AX$ on the interval $a < t < b$ and if $X_p(t)$ is any solution of the nonhomogeneous system $X' = AX + B(t)$ on the interval $a < t < b$, then any solution of the nonhomogeneous system can be written

$$X(t) = c_1 X_1(t) + \cdots + c_n X_n(t) + X_p(t)$$

for a unique choice of the constants $c_1, \ldots, c_n$.

Back to the example, if we pick $b_1 = b_2 = 1$, then

$$W\{X_1, X_2\} = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} e^{3t} = e^{3t} \neq 0$$

$\Rightarrow X_1, X_2$ linearly independent on any interval.

Thus, the general solution of (7) is

$$X(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}.$$

Theorem 11.7 If $m_1, m_2, \ldots, m_s$ are distinct eigenvalues of an $n \times n$ matrix A and if $X_1, X_2, \ldots, X_s$ are corresponding eigenvectors, then the set

$$\{X_1, \ldots, X_s\}$$

is linearly independent.

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2) $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$

Solution:

The characteristic equation of A is

$$\begin{vmatrix} 1-m & 0 \\ -2 & 2-m \end{vmatrix} = 2 - 2m - m + m^2 = m^2 - 3m + 2 = 0$$

$\Rightarrow m_1 = 1, \quad m_2 = 2$

For $m_1 = 1$,

$$\begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = O \Rightarrow -2c_1 + c_2 = 0 \Rightarrow c_2 = 2c_1$$

Let $c_1 = 1$. Then we get the eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t$ is a solution.

For $m_2 = 2$,

$$\begin{pmatrix} -1 & 0 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = O \Rightarrow c_1 = 0,\ c_2 \text{ any real, say } 1$$

$$\Rightarrow \text{eigenvector } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \text{ is a solution.}$$

$$W\{X_1, X_2\} = \begin{vmatrix} e^t & 0 \\ 2e^t & e^{2t} \end{vmatrix} = e^{3t} \neq 0 \Rightarrow X_1, X_2 \text{ lin. ind.}$$

Thus, the general solution is

$$X = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

4) $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

Solution:

Characteristic equation is

$$\begin{vmatrix} 1-m & 2 & -1 \\ 0 & -1-m & 3 \\ 0 & 0 & 2-m \end{vmatrix} = (1-m)\begin{vmatrix} -1-m & 3 \\ 0 & 2-m \end{vmatrix}$$

$$= (1-m)(-1-m)(2-m) = 0$$

$$\Rightarrow m_1 = 1,\quad m_2 = -1,\quad m_3 = 2$$

For $m_1 = 1$:

$$\begin{pmatrix} 0 & 2 & -1 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{r} 2c_2 - c_3 = 0 \\ -2c_2 + 3c_3 = 0 \\ c_3 = 0 \end{array}\right\} c_2 = c_3 = 0,\ c_1 \text{ arbitrary}$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t$$

For $m_2 = -1$:

$$\begin{pmatrix} 2 & 2 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} 2c_1 + 2c_2 - c_3 = 0 \\ 3c_3 = 0 \\ 3c_3 = 0 \end{matrix}\right\} c_3 = 0,\ 2c_1 + 2c_2 = 0 \Rightarrow c_1 = -c_2$$

$$\Rightarrow X_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-t}$$

For $m_3 = 2$:

$$\begin{pmatrix} -1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} -c_1 + 2c_2 - c_3 = 0 \\ -3c_2 + 3c_3 = 0 \end{matrix}\right\} c_2 = c_3,\ c_1 = c_3$$

$$\Rightarrow X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}$$

To establish linear independence of X_1, X_2, X_3:

$$W\{X_1(t), X_2(t), X_3(t)\} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} e^{2t} = e^{-2t} \neq 0\ .$$

Thus, the general solution is

$$X(t) = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-t} + c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}$$

11.6 Complex Eigenvalues

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2) $A = \begin{pmatrix} 4 & 1 \\ -8 & 8 \end{pmatrix}$

Solution:

$$\begin{vmatrix} 4-m & 1 \\ -8 & 8-m \end{vmatrix} = (4-m)(8-m)+8 = 32-12m+m^2+8$$

$$\Rightarrow m^2-12m+40=0$$

$$\Rightarrow m = \frac{12 \pm \sqrt{144-4(40)}}{2} = \frac{12 \pm \sqrt{-16}}{2} = 6 \pm 2i$$

For $m_1 = 6+2i$:

$$\begin{pmatrix} -2-2i & 1 \\ -8 & 2-2i \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = O$$

$$(-2-2i)c_1 + c_2 = 0 \Rightarrow c_2 = (2+2i)c_1$$

$$-8c_1 + (2-2i)c_2 = 0 \Rightarrow -8c_1 + (2-2i)(2+2i)c_1 = 0$$

$$\Rightarrow -8c_1 + (4-4i^2)c_1 = 0$$

$$\Rightarrow -8c_1 + 8c_1 = 0 \Rightarrow 0 = 0$$

$$\Rightarrow \text{eigenvector } \begin{pmatrix} 1 \\ 2+2i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} i$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 2+2i \end{pmatrix} e^{(6+2i)t}$$

For $m_1 = 6-2i$:

$$\begin{pmatrix} -2+2i & 1 \\ -8 & 2+2i \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = O$$

$$(-2+2i)c_1 + c_2 = 0 \Rightarrow c_2 = (2-2i)c_1$$

$$-8c_1 + (2+2i)c_2 = 0 \Rightarrow -8c_1 + (2+2i)(2-2i)c_1 = 0$$

$$\Rightarrow -8c_1 + (4-4i^2)c_1 = 0$$

$$\Rightarrow -8c_1 + 8c_1 = 0 \Rightarrow 0 = 0$$

$$\Rightarrow \text{eigenvector } \begin{pmatrix} 1 \\ 2-2i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} i$$

$$\Rightarrow X_2 = \begin{pmatrix} 1 \\ 2-2i \end{pmatrix} e^{(6-2i)t}$$

Thus,

$$X = c_1 \begin{pmatrix} 1 \\ 2+2i \end{pmatrix} e^{(6+2i)t} + c_2 \begin{pmatrix} 1 \\ 2-2i \end{pmatrix} e^{(6-2i)t}$$

Using Euler's formula $e^{(a+bi)t} = e^{at}(\cos bt + i \sin bt)$, we have

$$X = c_1\begin{pmatrix}1\\2+2i\end{pmatrix}e^{6t}(\cos 2t + i\sin 2t) + c_2\begin{pmatrix}1\\2-2i\end{pmatrix}e^{6t}(\cos 2t - i\sin 2t)$$

$$X = e^{6t}\left[(c_1+c_2)\begin{pmatrix}\cos 2t\\2\cos 2t - 2\sin 2t\end{pmatrix} + i(c_1-c_2)\begin{pmatrix}\sin 2t\\2\cos 2t + 2\sin 2t\end{pmatrix}\right]$$

$$X = e^{6t}\left[b_1\left\{\begin{pmatrix}1\\2\end{pmatrix}\cos 2t - \begin{pmatrix}0\\2\end{pmatrix}\sin 2t\right\} + b_2\left\{\begin{pmatrix}0\\2\end{pmatrix}\cos 2t + \begin{pmatrix}1\\2\end{pmatrix}\sin 2t\right\}\right]$$

The linear independence can be verified by computing $W(0)$:

$$W(0) = \begin{vmatrix}1 & 0\\2 & 2\end{vmatrix} = 2 \neq 0$$

Note:

1. The eigenvalues occur in conjugate pairs.
2. The eigenvalues corresponding to conjugate eigenvalues are also conjugates of one another.
3. $B = \begin{pmatrix}1\\2+2i\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix}i = \operatorname{Re} B + i\operatorname{Im} B$

   appears in the solution as

   $X = e^{6t}\left[b_1\{\operatorname{Re} B\cos 2t - \operatorname{Im} B\sin 2t\} + b_2\{\operatorname{Im} B\cos 2t + \operatorname{Re} B\sin 2t\}\right]$
4. $W(0) = |\operatorname{Re} B \cdot \operatorname{Im} B|$

Exercise #6 p. 208 $A = \begin{pmatrix}8 & -5\\16 & -8\end{pmatrix}$

Solution:

$$\begin{vmatrix}8-m & -5\\16 & -8-m\end{vmatrix} = -(8-m)(8+m) + 80 = m^2 + 16 = 0 \Rightarrow m = \pm 4i$$

For $m = 4i$

$$\begin{pmatrix}8-4i & -5\\16 & -8-4i\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$(8-4i)c_1 - 5c_2 = 0 \Rightarrow 5c_2 = (8-4i)c_1 \Rightarrow c_2 = \frac{(8-4i)c_1}{5}$$

$$16c_1 - (8+4i)c_2 = 0$$

Let $c_1 = 5$. Then $c_2 = 8-4i$. Eigenvector is $\begin{pmatrix}5\\8-4i\end{pmatrix} = \begin{pmatrix}5\\8\end{pmatrix} + \begin{pmatrix}0\\-4\end{pmatrix}i$

$$\Rightarrow X_1 = \begin{pmatrix}5\\8-4i\end{pmatrix}e^{4it} = \begin{pmatrix}5\\8-4i\end{pmatrix}(\cos 4t + i\sin 4t)$$

Similarly,

$$X_2 = \begin{pmatrix}5\\8+4i\end{pmatrix}e^{-4it} = \begin{pmatrix}5\\8-4i\end{pmatrix}(\cos 4t - i\sin 4t)$$

$$X = b_1\left\{\begin{pmatrix}5\\8\end{pmatrix}\cos 4t - \begin{pmatrix}0\\-4\end{pmatrix}\sin 4t\right\} + b_2\left\{\begin{pmatrix}0\\-4\end{pmatrix}\cos 4t + \begin{pmatrix}5\\8\end{pmatrix}\sin 4t\right\}$$

$$W = \begin{vmatrix}5 & 0\\8 & -4\end{vmatrix} = -20 \neq 0$$

11.7 Repeated Eigenvalues

Example 11.15 p. 208

Solve $X' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} X$. (1)

Solution:

The characteristic equation is

$$\begin{vmatrix} -m & 1 \\ -4 & 4-m \end{vmatrix} = 0 \Rightarrow (m-2)^2 = 0$$

For $m_1 = 2$,

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -2c_1 + c_2 = 0 \\ -4c_1 + 2c_2 = 0 \end{array}\right\} \Rightarrow c_2 = 2c_1$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} \qquad (2)$$

For $m_2 = 2$, guess...

$$X_2 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} te^{2t} \qquad (3)$$

But substituting (3) into (1) yields only the trivial solution $c_1 = c_2 = 0$:

$$X_2' = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{2t} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} 2te^{2t}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{2t} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} 2te^{2t} = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} te^{2t}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} 2t = \begin{pmatrix} c_2 \\ -4c_1 + 4c_2 \end{pmatrix} t$$

$$c_1(1+2t) = c_2 t \Rightarrow c_1(1+2t) - c_2 t = 0$$

$$c_2(1+2t) = (-4c_1 + 4c_2)t \Rightarrow 4c_1 t + (1-2t)c_2 = 0$$

$$\begin{vmatrix} 1+2t & -t \\ 4t & 1-2t \end{vmatrix} = 1 - 4t^2 + 4t^2 = 1$$

$$c_1 = \begin{vmatrix} 0 & -t \\ 0 & 1-2t \end{vmatrix} = 0; \quad c_2 = \begin{vmatrix} 1+2t & 0 \\ 4t & 0 \end{vmatrix} = 0$$

So try

$$X_2 = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} e^{2t} \qquad (4)$$

which is variation of parameters. Substitute (4) into (1):

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} 2e^{2t} + \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} e^{2t} = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} e^{2t}$$

OR

$$\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \qquad (5)$$

which can be written

$$\left.\begin{aligned} c_1'(t) &= -2c_1(t) + c_2(t) \\ c_2'(t) &= -4c_1(t) + 2c_2(t) = 2c_1'(t) \end{aligned}\right\} \qquad (6)$$

$\Rightarrow c_2(t) = 2c_1(t) + a,$ a arbitrary constant

Substitute this in the 1st equation of (6):

$$c_1'(t) = -2c_1(t) + 2c_1(t) + a$$

$$c_1'(t) = a$$

$c_1(t) = at + b \Rightarrow$ solution set of (5) is

$$c_1(t) = at + b$$

$$c_2(t) = 2at + 2b + a$$

So (4) becomes

$$X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} ate^{2t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} be^{2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} ae^{2t}$$

Note if we choose $a = 0,\ b = 1$ we get X_1, so choose $a = 1,\ b = 0$ to get

$$X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}.$$

At $t = 0$,

$$W(X_1, X_2) = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow X_1, X_2$ linearly independent

$$X = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \right]$$

Ex 11.16 p. 210 Solve $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 8 & -1 \\ 4 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (7)$

Solution:

$$\begin{vmatrix} 8-m & -1 \\ 4 & 12-m \end{vmatrix} = 0 \Rightarrow (m-10)^2 = 0$$

For $m_1 = 10$,

$$\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -2c_1 - c_2 = 0 \\ 4c_1 + 2c_2 = 0 \end{array}\right\} \Rightarrow c_2 = 2c_1$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{10t} \qquad (8)$$

Now, find

$$X_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} te^{10t} + \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} e^{10t} \qquad (9)$$

Substitute (9) into (7):

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} 10te^{10t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{10t} + \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} 10e^{10t} = \begin{pmatrix} 8 & -1 \\ 4 & 12 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} te^{10t} + \begin{pmatrix} 8 & -1 \\ 4 & 12 \end{pmatrix}\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} e^{10t}$$

$$\Rightarrow \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} e^{10t} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{10t}$$

or $\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\left.\begin{array}{l} -2c_3 - c_4 = 1 \\ 4c_3 + 2c_4 = -2 \end{array}\right\} \Rightarrow c_4 = -1 - 2c_3; \;\; c_3 = 0, \; c_4 = -1$$

$$X_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} te^{10t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{10t}$$

$$X = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{10t} + c_2 e^{10t}\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

Ex 11.17 p. 211

Solve

$$\begin{aligned} &D^2 y + (D-1)v = 0 \\ &(2D-1)y + (D-1)w = 0 \qquad\qquad (11) \\ &(D+3)y + (D-4)v + 3w = 0 \end{aligned}$$

Solution:

Reduce (11) to a system of 1st-order equations.
Let $Dy = u$. Then (11) can be written

$$\begin{aligned} Du &= u - 3v + 3w + 3y \\ Dv &= -u + 4v - 3w - 3y \\ Dw &= -2u \qquad + w + y \qquad\qquad (12) \\ Dy &= u \end{aligned}$$

$$A=\begin{pmatrix}1 & -3 & 3 & 3\\ -1 & 4 & -3 & -3\\ -2 & 0 & 1 & 1\\ 1 & 0 & 0 & 0\end{pmatrix}$$

$$|A-mI|=\begin{vmatrix}1-m & -3 & 3 & 3\\ -1 & 4-m & -3 & -3\\ -2 & 0 & 1-m & 1\\ 1 & 0 & 0 & -m\end{vmatrix}=0$$

Expand about the 4th row:

$$|A-mI|=-\begin{vmatrix}-3 & 3 & 3\\ 4-m & -3 & -3\\ 0 & 1-m & 1\end{vmatrix}-m\begin{vmatrix}1-m & -3 & 3\\ -1 & 4-m & -3\\ -2 & 0 & 1-m\end{vmatrix}$$

$$=-(-3)\left[-3+3(1-m)\right]+3(4-m)-3(4-m)-m(4-m)(1-m)-3m\left[(m-1)-6\right]-3m\cdot 2(4-m)$$

$$=-9m-m\left(4-5m+m^2\right)-3m^2+21m-24m+6m^2$$

$$=$$

Next we look at an example in which a repeated root gives rise to two linearly independent eigenvectors of the matrix:

Ex 11.18 p. 212

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}'=\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} \qquad (13)$$

$$|A-mI|=\begin{vmatrix}-m & 1 & 1\\ 1 & -m & 1\\ 1 & 1 & -m\end{vmatrix}=m^3-3m-2=0$$

$$(m+1)^2(m-2)=0\Rightarrow m=-1,-1,2$$

For $m=-1,$ solve

$$\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow c_1+c_2+c_3=0$$

$$\Rightarrow\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}=\begin{pmatrix}c_1\\ c_2\\ -c_1-c_2\end{pmatrix}=c_1\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}+c_2\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}e^{-t}\text{ and }\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}e^{-t}\text{ are solutions of }(13)$$

For $m=2$, solve

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -2c_1+c_2+c_3=0$$

$$c_1-2c_2+c_3=0$$

$$c_1+c_2-2c_3=0$$

$$\Rightarrow c_1=c_3 \text{ and } c_2=c_3$$

$$\Rightarrow C=c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{2t} \text{ is the 3rd sol of } (13)$$

W at $t=0$ is

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix}=3\neq 0$$

Gen sol is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{-t}+c_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}e^{-t}+c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{2t}$$

Exercises

1) $A=\begin{pmatrix} 4 & 1 \\ -4 & 8 \end{pmatrix}$

Solution:

$$\begin{vmatrix} 4-m & 1 \\ -4 & 8-m \end{vmatrix}=(4-m)(8-m)+4=m^2-12m+36=0$$

$$\Rightarrow m=6,6$$

For $m=6$, solve

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\Rightarrow \begin{matrix} -2c_1+c_2=0 \\ -4c_1+2c_2=0 \end{matrix}\Rightarrow 2c_1=c_2$$

$$\Rightarrow X_1=\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{6t}$$

Let $X_2=\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}e^{6t}$

$$\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} e^{6t} + 6\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} e^{6t} = \begin{pmatrix} 4 & 1 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} e^{6t}$$

$$\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} + 6\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

$$c_1'(t) + 6c_1(t) = 4c_1(t) + c_2(t) \Rightarrow c_1'(t) = -2c_1(t) + c_2(t)$$

$$c_2'(t) + 6c_2(t) = -4c_1(t) + 8c_2(t) \Rightarrow c_2'(t) = -4c_1(t) + 2c_2(t)$$

$$c_2'(t) = 2c_1'(t) \Rightarrow c_2(t) = 2c_1(t) + a$$

$$\Rightarrow c_1'(t) = -2c_1(t) + 2c_1(t) + a = a$$

$$\Rightarrow c_1(t) = at + b$$

Thus, we have

$c_1(t) = at + b$ and $c_2(t) = 2at + 2b + a$

$$X_2 = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} e^{6t} = \begin{pmatrix} at + b \\ 2at + 2b + a \end{pmatrix} e^{6t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} ate^{6t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} be^{6t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} ae^{6t}$$

Choose $a = 1$ and $b = 0$

$$X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{6t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{6t}$$

$$W = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

2) $A = \begin{pmatrix} 4 & -9 \\ 4 & -8 \end{pmatrix}$

Solution:

$$\begin{vmatrix} 4 - m & -9 \\ 4 & -8 - m \end{vmatrix} = -(4 - m)(8 + m) + 36 = m^2 + 4m + 4 = 0$$

$$\Rightarrow m = -2, -2$$

For $m = -2$, solve

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 6c_1 - 9c_2 = 0 \\ 4c_1 - 6c_2 = 0 \end{matrix} \Rightarrow 2c_1 - 3c_2 = 0$$

$$\Rightarrow c_2 = \frac{2}{3}c_1 \Rightarrow c_1 = 3, c_2 = 2$$

$$X_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-2t}$$

Now find

$$X_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} te^{-2t} + \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} e^{-2t}$$

Substitute:

$$\begin{pmatrix}3\\2\end{pmatrix}e^{-2t}-2\begin{pmatrix}3\\2\end{pmatrix}te^{-2t}-2\begin{pmatrix}c_3\\c_4\end{pmatrix}e^{-2t}=\begin{pmatrix}4&-9\\4&-8\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}te^{-2t}+\begin{pmatrix}4&-9\\4&-8\end{pmatrix}\begin{pmatrix}c_3\\c_4\end{pmatrix}e^{-2t}$$

$$\begin{pmatrix}3\\2\end{pmatrix}-2\begin{pmatrix}3\\2\end{pmatrix}t-2\begin{pmatrix}c_3\\c_4\end{pmatrix}=\begin{pmatrix}4&-9\\4&-8\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}t+\begin{pmatrix}4&-9\\4&-8\end{pmatrix}\begin{pmatrix}c_3\\c_4\end{pmatrix}$$

$$\begin{pmatrix}3-2c_3\\2-2c_4\end{pmatrix}=\begin{pmatrix}4c_3-9c_4\\4c_3-8c_4\end{pmatrix}$$

$$-6c_3+9c_4=-3\Rightarrow -2c_3+3c_4=-1\Rightarrow 3c_4=-1+2c_3$$

$$-4c_3+6c_4=-2\Rightarrow -2c_3+3c_4=-1$$

$$\begin{pmatrix}c_3\\c_4\end{pmatrix}=\begin{pmatrix}1/2\\0\end{pmatrix}$$

$$X_2=\begin{pmatrix}3\\2\end{pmatrix}te^{-2t}+\begin{pmatrix}1/2\\0\end{pmatrix}e^{-2t}\quad\text{or use } X_2=\begin{pmatrix}6\\4\end{pmatrix}te^{-2t}+\begin{pmatrix}1\\0\end{pmatrix}e^{-2t}$$

3) $A=\begin{pmatrix}2&-1\\4&6\end{pmatrix}$

Solution:

$$\begin{vmatrix}2-m&-1\\4&6-m\end{vmatrix}=(2-m)(6-m)+4=m^2-8m+16=0$$

$$\Rightarrow m=4,4$$

For $m=4$, solve

$$\begin{pmatrix}-2&-1\\4&2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow \begin{matrix}-2c_1-c_2=0\\4c_1+2c_2=0\end{matrix}\Rightarrow -2c_1=c_2$$

$$\Rightarrow X_1=\begin{pmatrix}1\\-2\end{pmatrix}e^{4t}$$

Let $X_2=\begin{pmatrix}1\\-2\end{pmatrix}te^{4t}+\begin{pmatrix}c_3\\c_4\end{pmatrix}e^{4t}$

$$4\begin{pmatrix}1\\-2\end{pmatrix}te^{4t}+\begin{pmatrix}1\\-2\end{pmatrix}e^{4t}+4\begin{pmatrix}c_3\\c_4\end{pmatrix}e^{4t}=\begin{pmatrix}2&-1\\4&6\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}te^{4t}+\begin{pmatrix}2&-1\\4&6\end{pmatrix}\begin{pmatrix}c_3\\c_4\end{pmatrix}e^{4t}$$

$$\begin{pmatrix}1\\-2\end{pmatrix}+4\begin{pmatrix}c_3\\c_4\end{pmatrix}=\begin{pmatrix}2&-1\\4&6\end{pmatrix}\begin{pmatrix}c_3\\c_4\end{pmatrix}$$

$$1+4c_3=2c_3-c_4\Rightarrow 2c_3+c_4=-1\Rightarrow c_4=-1-2c_3$$

$$-2+4c_4=4c_3+6c_4\Rightarrow -4c_3-2c_4=2$$

$$\begin{pmatrix}c_3\\c_4\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}$$

$$X_2=\begin{pmatrix}1\\-2\end{pmatrix}te^{4t}+\begin{pmatrix}0\\-1\end{pmatrix}e^{4t}$$

$$X=\begin{pmatrix}1\\-2\end{pmatrix}e^{4t}+\begin{pmatrix}1\\-2\end{pmatrix}te^{4t}+\begin{pmatrix}0\\-1\end{pmatrix}e^{4t}$$

4) $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$

Solution:

$$\begin{vmatrix} 1-m & -2 \\ 2 & -3-m \end{vmatrix} = -(1-m)(3+m)+4 = m^2+2m+1 = 0$$

$\Rightarrow m = -1, -1$

For $m = -1$, solve

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 2c_1 - 2c_2 = 0 \\ 2c_1 - 2c_2 = 0 \end{matrix} \Rightarrow c_1 = c_2$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Let $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} e^{-t}$

$$-\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} - \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} e^{-t} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} e^{-t}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

$1 - c_3 = c_3 - 2c_4 \Rightarrow -2c_3 + 2c_4 = -1 \Rightarrow 2c_4 = -1 + 2c_3$

$1 - c_4 = 2c_3 - 3c_4 \Rightarrow -2c_3 + 2c_4 = -1$

$$\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} e^{-t}$ or use $X_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{-t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$

$$X = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{-t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$$

6) $A=\begin{pmatrix}2&1&-1\\0&-1&2\\0&0&-1\end{pmatrix}$

Solution:

$$|A-mI|=\begin{vmatrix}2-m&1&-1\\0&-1-m&2\\0&0&-1-m\end{vmatrix}=(2-m)(-1-m)^2=0$$

$\Rightarrow m=-1,-1,2$

For $m=-1$, solve

$$\begin{pmatrix}3&1&-1\\0&0&2\\0&0&0\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow \begin{matrix}3c_1+c_2-c_3=0\\2c_3=0\end{matrix}$$

$$\Rightarrow\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}c_1\\-3c_1\\0\end{pmatrix}=c_1\begin{pmatrix}1\\-3\\0\end{pmatrix}$$

$$\Rightarrow X_1=\begin{pmatrix}1\\-3\\0\end{pmatrix}e^{-t}$$

Let $X_2=\begin{pmatrix}1\\-3\\0\end{pmatrix}te^{-t}+\begin{pmatrix}c_4\\c_5\\c_6\end{pmatrix}e^{-t}$

$$-\begin{pmatrix}1\\-3\\0\end{pmatrix}te^{-t}+\begin{pmatrix}1\\-3\\0\end{pmatrix}e^{-t}-\begin{pmatrix}c_4\\c_5\\c_6\end{pmatrix}e^{-t}=\begin{pmatrix}2&1&-1\\0&-1&2\\0&0&-1\end{pmatrix}\begin{pmatrix}1\\-3\\0\end{pmatrix}te^{-t}+\begin{pmatrix}2&1&-1\\0&-1&2\\0&0&-1\end{pmatrix}\begin{pmatrix}c_4\\c_5\\c_6\end{pmatrix}e^{-t}$$

$$\begin{pmatrix}1\\-3\\0\end{pmatrix}-\begin{pmatrix}c_4\\c_5\\c_6\end{pmatrix}=\begin{pmatrix}2&1&-1\\0&-1&2\\0&0&-1\end{pmatrix}\begin{pmatrix}c_4\\c_5\\c_6\end{pmatrix}$$

$1-c_4=2c_4+c_5-c_6\Rightarrow 3c_4+c_5-c_6=1\Rightarrow 3c_4+c_5=-1/2\Rightarrow 6c_4+2c_5=-1$

$-3-c_5=-c_5+2c_6\Rightarrow c_6=-3/2$

$-c_6=-c_6$

$$\begin{pmatrix}c_4\\c_5\\c_6\end{pmatrix}=\begin{pmatrix}-1/6\\0\\-3/2\end{pmatrix}$$

$X_2=\begin{pmatrix}1\\-3\\0\end{pmatrix}te^{-t}+\begin{pmatrix}-1/6\\0\\-3/2\end{pmatrix}e^{-t}$ or use $X_2=\begin{pmatrix}6\\-18\\0\end{pmatrix}te^{-t}+\begin{pmatrix}-1\\0\\-9\end{pmatrix}e^{-t}$

For $m=2$, solve

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} c_2 - c_3 = 0 \\ -3c_2 + 2c_3 = 0 \Rightarrow c_2 = 0 \\ -3c_3 = 0 \Rightarrow c_3 = 0 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$\Rightarrow X_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$$

Thus,

$$X = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 6 \\ -18 \\ 0 \end{pmatrix} te^{-t} + \begin{pmatrix} -1 \\ 0 \\ -9 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}.$$

11.5 Review

6) $A = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$

Solution:

$$|A - mI| = \begin{vmatrix} 2-m & 3 \\ 1 & -2-m \end{vmatrix} = (2-m)(-2-m) - 3$$

$$m^2 - 7 = 0 \Rightarrow m = \pm\sqrt{7}$$

For $m = \sqrt{7}$, solve

$$\begin{pmatrix} 2-\sqrt{7} & 3 \\ 1 & -2-\sqrt{7} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(2-\sqrt{7}\right)c_1 + 3c_2 = 0 \Rightarrow c_1 = -\frac{3}{2-\sqrt{7}}c_2$$

$$c_1 - \left(2+\sqrt{7}\right)c_2 = 0 \Rightarrow c_1 = \left(2+\sqrt{7}\right)c_2$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2-\sqrt{7} \end{pmatrix}$$

$$X_1 = \begin{pmatrix} -3 \\ 2-\sqrt{7} \end{pmatrix} e^{\sqrt{7}t}$$

For $m = -\sqrt{7}$, solve

$$\begin{pmatrix} 2+\sqrt{7} & 3 \\ 1 & -2+\sqrt{7} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(2+\sqrt{7}\right)c_1 + 3c_2 = 0 \Rightarrow c_1 = -\frac{3}{2+\sqrt{7}}c_2$$

$$c_1 - \left(2-\sqrt{7}\right)c_2 = 0 \Rightarrow c_1 = \left(2-\sqrt{7}\right)c_2$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2+\sqrt{7} \end{pmatrix}$$

$$X_2 = \begin{pmatrix} -3 \\ 2+\sqrt{7} \end{pmatrix} e^{-\sqrt{7}t}$$

$$W\left(X_1(0), X_2(0)\right) = \begin{vmatrix} -3 & -3 \\ 2-\sqrt{7} & 2+\sqrt{7} \end{vmatrix} = -6 - 3\sqrt{7} + 6 - 3\sqrt{7} \neq 0$$

$$X = c_1 \begin{pmatrix} -3 \\ 2-\sqrt{7} \end{pmatrix} e^{\sqrt{7}t} + c_2 \begin{pmatrix} -3 \\ 2+\sqrt{7} \end{pmatrix} e^{-\sqrt{7}t}$$

11.6 Review

6) $A = \begin{pmatrix} 8 & -5 \\ 16 & -8 \end{pmatrix}$

Solution:

$$|A - mI| = \begin{vmatrix} 8-m & -5 \\ 16 & -8-m \end{vmatrix} = (8-m)(-8-m) + 80$$

$m^2 + 16 = 0 \Rightarrow m = \pm 4i$

For $m = 4i$, solve

$$\begin{pmatrix} 8-4i & -5 \\ 16 & -8-4i \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(8-4i)c_1 - 5c_2 = 0 \Rightarrow c_1 = \frac{5}{8-4i}c_2$$

$$16c_1 - (8+4i)c_2 = 0 \Rightarrow c_1 = \frac{8+4i}{16}c_2$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 8-4i \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix} i$$

Thus,

$$X = e^{0t}\left[c_1\{\operatorname{Re} B \cos 4t - \operatorname{Im} B \sin 4t\} + c_2\{\operatorname{Im} B \cos 4t + \operatorname{Re} B \sin 4t\}\right]$$

$$X = c_1\left\{\begin{pmatrix} 5 \\ 8 \end{pmatrix}\cos 4t - \begin{pmatrix} 0 \\ -4 \end{pmatrix}\sin 4t\right\} + c_2\left\{\begin{pmatrix} 0 \\ -4 \end{pmatrix}\cos 4t + \begin{pmatrix} 5 \\ 8 \end{pmatrix}\sin 4t\right\}$$