MATH 267 CHAPTER 17 MULTIPLE INTEGRALS

17.9 CHANGE OF VARIABLES AND JACOBIANS

This section talks of a more general method for changing variables in multiple integrals.

Consider a function $T : D \to E$ whose domain *D* is a region in the *xy*-plane and whose range *E* is a region in the *uv*-plane. To each point (x, y) in *D* there corresponds exactly one point (u,v) in E such that

$$T(x, y) = (u,v) .$$

T is called a **transformation of coordinates** from the *xy*-plane to the *uv*-plane. Since each pair (u,v) is uniquely determined by (x, y), both *u* and *v* are functions of *x* and *y*. Thus, we have the following transformation formulas:

$$u = f(x, y), \quad v = g(x, y); \quad (x, y) \text{ in } D, \quad (u,v) \text{ in } E$$

If *T* is one-to-one, we obtain the **inverse** T^{-1} from the *uv*-plane to the *xy*-plane. T^{-1} may be specified by

$$x = F(u,v), \qquad y = G(u,v)$$

for certain functions *F* and *G*.

Exercise

Given $T : (x, y) \to (u,v)$

a) Describe the *u*-curves and *v*-curves.

b) Find formulas for $x = F(u,v)$, $y = G(u,v)$ that specify T^{-1}.

2) $u = \frac{1}{2}y$, $v = \frac{1}{3}x$

Solution:

$u = \frac{1}{2}y,$	$v = \frac{1}{3}x$
$u = 1 \Rightarrow y = 2$	$v = 1 \Rightarrow x = 3$
$u = 2 \Rightarrow y = 4$	$v = 2 \Rightarrow x = 6$
$u = 3 \Rightarrow y = 6$	$v = 3 \Rightarrow x = 9$
horizontal lines	vertical lines

b) $x = 3v$, $y = 2u$

4) $u = -5x + 4y$, $v = 2x - 3y$

Solution:

$u = -5x + 4y,$	$v = 2x - 3y$
$u = 1 \Rightarrow -5x + 4y = 1$	$v = 1 \Rightarrow 2x - 3y = 1$
$u = 2 \Rightarrow -5x + 4y = 2$	$v = 2 \Rightarrow 2x - 3y = 2$
$u = 3 \Rightarrow -5x + 4y = 3$	$v = 3 \Rightarrow 2x - 3y = 3$
lines with slope $5/4$	lines with slope $2/3$

$$u = -5x + 4y \xrightarrow{\times 3} 3u = -15x + 12y$$
$$v = 2x - 3y \xrightarrow{\times 4} \underline{4v = \quad 8x - 12y}$$
$$3u + 4v = -7x \Rightarrow x = \frac{3u+4v}{-7} \Rightarrow x = -\frac{3}{7}u - \frac{4}{7}v$$

$$u = -5x + 4y \xrightarrow{\times 2} 2u = -10x + 8y$$
$$v = 2x - 3y \xrightarrow{\times 5} \underline{5v = \quad 10x - 15y}$$
$$2u + 5v = -7y \Rightarrow y = \frac{2u+5v}{-7} \Rightarrow y = -\frac{2}{7}u - \frac{5}{7}v$$

6) $u = x+1, \quad v = 2 - y^3$

Solution:

$u = x+1,$	$v = 2 - y^3$
$x+1 = 1 \Rightarrow x = 0$	$2 - y^3 = 2 \Rightarrow y = 0$
$x+1 = 3 \Rightarrow x = 2$	$2 - y^3 = 10 \Rightarrow y^3 = -8 \Rightarrow y = -2$
$x+1 = -1 \Rightarrow x = -2$	
vertical lines	horizontal lines

$u = x+1, \quad v = 2 - y^3$

$x = u - 1, \quad y = (2-v)^{1/3}$

8) $u = e^{2y}, \quad v = e^{-3x}$

Solution:

$u = e^{2y},$

$e^{2y} = 1 \Rightarrow 2y = \ln 1 \Rightarrow 2y = 0 \Rightarrow y = 0$

$e^{2y} = e \Rightarrow 2y = \ln e \Rightarrow 2y = 1 \Rightarrow y = 1/2$

horizontal lines

$v = e^{-3x}$

$e^{-3x} = 1 \Rightarrow -3x = \ln 1 = 0 \Rightarrow x = 0$

$e^{-3x} = e \Rightarrow -3x = \ln e = 1 \Rightarrow x = -1/3$

vertical lines

$u = e^{2y}$	$v = e^{-3x}$
$2y = \ln u$	$-3x = \ln v$
$y = \frac{1}{2}\ln u$	$x = -\frac{1}{3}\ln v$

Example 3 p. 951 Cartesian coordinates to Polar Coordinates

Finding the curve in the *uv*-plane that corresponds to the curve in the *xy*-plane:

10) $u = \frac{1}{2}y, \quad v = \frac{1}{3}x$

(a) triangle with vertices $(0,0),\ (3,6),\ (9,4)$

(b) the line $3x - 2y = 4$

Solution:

(a) triangle with vertices $(0,0),\ (3,6),\ (9,4)$

$$(0,0): \quad u = 0,\ v = 0 \Rightarrow (0,0)$$

$$(3,6): \quad u = \frac{6}{2} = 3,\ v = \frac{3}{3} = 1 \Rightarrow (3,1)$$

$$(9,4): \quad u = \frac{4}{2} = 2,\ v = \frac{9}{3} = 3 \Rightarrow (2,3)$$

(b) the line $3x - 2y = 4$

$$\left.\begin{aligned} u = \frac{1}{2}y \Rightarrow y = 2u \\ v = \frac{1}{3}x \Rightarrow x = 3v \end{aligned}\right\} \Rightarrow 3(3v) - 2(2u) = 4 \Rightarrow 9v - 4u = 4$$

12) $u = -5x + 4y, \quad v = 2x - 3y$

(a) the square with vertices $(0,0),\ (1,-1),\ (2,0),\ (1,1)$

(b) the circle $x^2 + y^2 = 1$

Solution:

(a) the square with vertices $(0,0),\ (1,-1),\ (2,0),\ (1,1)$

$$(0,0): \quad u = 0,\ v = 0 \Rightarrow (0,0)$$

$$(1,-1): \quad u = -5(1) + 4(-1) = -9,\ v = 2(1) - 3(-1) = 5 \Rightarrow (-9,5)$$

$$(2,0): \quad u = -5(2) + 4(0) = -10,\ v = 2(2) - 3(0) = 4 \Rightarrow (-10,4)$$

$$(1,1): \quad u = -5(1) + 4(1) = -1,\ v = 2(1) - 3(1) = -1 \Rightarrow (-1,-1)$$

(b) the circle $x^2 + y^2 = 1$

$$u = -5x + 4y \xrightarrow{\times 2} 2u = -10x + 8y$$

$$v = 2x - 3y \xrightarrow{\times 5} \underline{5v = \quad 10x - 15y}$$

$$2u + 5v = -7y \Rightarrow y = \frac{2u + 5v}{-7}$$

$$u = -5x + 4y \xrightarrow{\times 3} 3u = -15x + 12y$$

$$v = 2x - 3y \xrightarrow{\times 4} \underline{4v = \quad 8x - 12y}$$

$$3u + 4v = -7x \Rightarrow x = \frac{3u + 4v}{-7}$$

$$\left(\frac{3u+4v}{-7}\right)^2+\left(\frac{2u+5v}{-7}\right)^2=1$$

$$9u^2+24uv+16v^2+4u^2+20uv+25v^2=49$$

$$13u^2+44uv+41v^2=49$$

$$B^2-4AC=44^2-4(13)(41)<0 \Rightarrow \text{ellipse}$$

Changing Variables in Double Integration

$$\iint_R F(x,y)\,dA=\iint_S F\big(f(u,v),g(u,v)\big)\,dA$$

To do so, we need restrictions on R and $z=F(x,y)$:

1) Assume that R consists of all points that are either inside or on a piecewise-smooth simple closed curve C

2) Assume F has continuous first partial derivatives throughout an open region containing R.
   Positive direction along C: as P traces out C, R is on the left
   Negative direction along C: as P traces out C, R is on the right

3) W should transform a region S of the uv-plane in a one-to-one manner onto R.

4) S should be bounded by a piecewise-smooth simple closed curve K that W transforms onto C.

5) As (u,v) traces out K once in the positive direction, the corresponding point (x,y) traces C once in either the positive or the negative direction.

Definition If $x=f(u,v)$ and $y=g(u,v)$, then the **<u>Jacobian</u>** of x and y with respect to u and v is

Theorem If $x=f(u,v)$, $y=g(u,v)$ is a transformation of coordinates, then

$$\iint_R F(x,y)\,dx\,dy=\pm\iint_S F(f(u,v),g(u,v))\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv\,.$$

As (u,v) traces the boundary K of S once in the positive direction, the corresponding point (x,y) traces the boundary C of R once in either the positive direction (+ sign) or the negative direction (- sign).

Note If $F(x,y)=1$ for every (x,y) in R

$$\iint_R dx\,dy=\pm\iint_S \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

gives the area of R.
If $\dfrac{\partial(x,y)}{\partial(u,v)}>0$ then + is chosen.

Corollary If $x=f(u,v)$, $y=g(u,v)$ is a transformation of coordinates and if the Jacobian does not change sign in S, then

$$\iint_R F(x,y)\,dx\,dy = \iint_S F(f(u,v),g(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Exercises

Find the Jacobian.

14) $x = e^u \sin v, \quad y = e^u \cos v$

Solution:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \sin v & e^u \cos v \\ e^u \cos v & -e^u \sin v \end{vmatrix} = -e^{2u}\sin^2 v - e^{2u}\cos^2 v = -e^{2u}$$

16) $x = \frac{u}{u^2+v^2}, \quad y = \frac{v}{u^2+v^2}$

Solution:

$$\frac{\delta x}{\delta u} = \frac{(u^2+v^2)\cdot 1 - u\cdot 2u}{(u^2+v^2)^2} = \frac{v^2-u^2}{(u^2+v^2)^2} \qquad \frac{\delta x}{\delta v} = -\frac{2uv}{(u^2+v^2)^2}$$

$$\frac{\delta y}{\delta u} = -\frac{2uv}{(u^2+v^2)^2} \qquad \frac{\delta y}{\delta u} = \frac{(u^2+v^2)\cdot 1 - v\cdot 2v}{(u^2+v^2)^2} = \frac{u^2-v^2}{(u^2+v^2)^2}$$

$$\begin{aligned}\frac{\delta(x,y)}{\delta(u,v)} &= \frac{(v^2-u^2)(u^2-v^2)}{(u^2+v^2)^4} - \frac{4u^2v^2}{(u^2+v^2)^4} \\ &= \frac{u^2v^2-u^4-v^4+u^2v^2-4u^2v^2}{(u^2+v^2)^4} \\ &= -\frac{u^4+2u^2v^2-v^4}{(u^2+v^2)^4} \\ &= -\frac{1}{(u^2+v^2)^2}\end{aligned}$$

Changing the variables of integration:

20) $\iint_R (3x-4y)\,dxdy \quad R: \; y=3x, \quad y=\frac{1}{2}x, \quad x=4\,; \quad x=u-2v, \; y=3u-v$

Solution:

$$x = u-2v \longrightarrow \quad x = u-2v$$

$$y = 3u - v \xrightarrow{\times(-2)} \underline{-2y = -6u+2v}$$

$$x-2y = -5u \Rightarrow u = -\frac{1}{5}x + \frac{2}{5}y$$

$$3x = 3u - 6v$$
$$\underline{y = 3u - v}$$
$$3x - y = -5v \Rightarrow v = -\frac{3}{5}x + \frac{1}{5}y$$

$$y = 3x \Rightarrow 3u - v = 3(u - 2v)$$
$$3u - v = 3u - 6v$$
$$-v = -6v \Rightarrow v = 0$$

$$y = \frac{1}{2}x \Rightarrow 3u - v = \frac{1}{2}(u - 2v)$$
$$3u - v = \frac{1}{2}u - v$$
$$3u = \frac{1}{2}u \Rightarrow u = 0$$

$$x = 4 \Rightarrow u - 2v = 4$$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = -1 + 6 = 5$$

Thus,

$$\iint_R (3x - 4y)\,dxdy = \int_{-2}^{0}\int_{0}^{2v+4} \left[3(u - 2v) - 4(3u - v)\right] \cdot 5\ dudv$$
$$= \int_{-2}^{0}\int_{0}^{2v+4} (-9u - 2v) \cdot 5\ dudv$$

22) $\iint_R xy\ dx\ dy;\ \ R:\ \ y = 2\sqrt{1 - x},\ \ x = 0,\ \ y = 0;\ \ x = u^2 - v^2,\ \ y = 2uv$

Solution:

T is <u>not</u> 1 – 1 so we restrict x, $y > 0$

$$x = 0: \quad u^2 - v^2 = 0 \Rightarrow u^2 = v^2 \Rightarrow u = v$$

$y = 0: \quad 2uv = 0 \Rightarrow$ ~~$u = 0$~~ or $v = 0$

since $0 < u^2 - v^2 < 1$

$$y = 2\sqrt{1 - x}: \quad 2uv = 2\sqrt{1 - u^2 + v^2}$$
$$u^2v^2 = 1 - u^2 + v^2$$
$$u^2v^2 + u^2 - v^2 - 1 = 0$$
$$(u^2 - 1)(v^2 + 1) = 0 \Rightarrow u = \pm 1 \Rightarrow u = 1$$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$

$$\int_0^1\int_v^1 (u^2 - v^2)(2uv)(4u^2 + 4v^2)\,dudv$$

Extension to 3D

Definition If $x = f(u,v,w),\ y = g(u,v,w),$ and $z = h(u,v,w)$, then the **Jacobian** of *x, y,* and *z* with respect to *u, v,* and *w* is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Theorem If $x = f(u,v,w),\ y = g(u,v,w),\ z = h(u,v,w)$ is a transformation of coordinates from a region *S* in a *uvw*-coordinate system onto a region *R* in an *xyz*-coordinate system and if the Jacobian does not change sign in *S*, then

$$\iiint_R F(x,y,z)\,dx\,dy\,dz = \iiint_S G(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw\,,$$

where $G(u,v,w)$ is the expression obtained by substituting for *x, y*, and *z* in $F(x,y,z)$.

Exercises

34) Derive a formula for $\iiint_R F(x,y,z)\,dxdydz$ for a transformation from rectangular to cylindrical coordinates.

Solution:

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$$

$$J = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta\left(r\cos\theta\right) + r\sin\theta\left(\sin\theta\right) + 0 = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

$$\iiint_S G(r,\theta,z)\ r\,dr\,d\theta\,dz$$

Ex… cylindrical coordinates

Solution:

$$x = \rho\sin\phi\cos\theta,\quad y = \rho\sin\phi\sin\theta,\quad z = \rho\cos\phi$$

$$J=\begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \sin\phi\cos\theta\left(\rho^2\sin^2\phi\cos\theta\right) - \rho\cos\phi\cos\theta\left(-\rho\sin\phi\cos\theta\cos\phi\right)$$

$$-\rho\sin\phi\sin\theta\left(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right)$$

$$= \rho^2\sin^3\phi\cos^2\theta + \rho^2\sin\phi\cos^2\phi + \rho^2\sin^2\phi\sin^2\theta$$

$$= \rho^2\sin\phi\left(\sin^2\phi\cos^2\theta + \cos^2\phi\cos^2\theta + \sin^2\theta\right)$$

$$= \ldots$$

24) $\iint_R \sin\frac{y-x}{y+x}\,dxdy$; R: the trapezoid with vertices $(1,1),\ (2,2),(4,0),(2,0)$

$T:\quad u = y - x,\quad v = y + x$

Solution:

$(1,1):\quad u = 0,\ v = 2$

$(2,2):\quad u = 0,\ v = 4$

$(4,0):\quad u = -4,\ v = 4$

$(2,0):\quad u = -2,\ v = 2$

$$\left.\begin{array}{l} u = y - x \\ v = y + x \end{array}\right\} \Rightarrow u + v = 2y \Rightarrow y = \frac{1}{2}(u+v)$$

$$\left.\begin{array}{l} u = y - x \\ -v = -y - x \end{array}\right\} \Rightarrow u - v = -2x \Rightarrow x = -\frac{1}{2}(u-v)$$

$$J = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$-\int_2^4\int_{-v}^0 \sin\frac{u}{v}\left(-\frac{1}{2}\right)dudv$$

$$= \frac{1}{2}\int_2^4 \left(-\cos\frac{u}{v}\right)v\Big]_{-v}^0 dv$$

$$= \frac{1}{2}\int_2^4 \left[(-\cos 0)v + \left(\cos(-1)v\right)\right]dv$$

$$= \frac{1}{2}\int_2^4 \left[-v + (\cos 1)v\right]dv = \frac{1}{2}\left[-\frac{v^2}{2} + (\cos 1)\frac{v^2}{2}\right]_2^4$$

$$= \frac{1}{2}\left[-8 + (\cos 1)8 + 2 - (\cos 1)2\right] = \frac{1}{2}(-6 + 6\cos 1) = 3\cos 1 - 3$$

26) $\iint_R \frac{1}{(4x-4y+1)^2}\,dxdy$; $R:\quad x = \sqrt{-y},\quad x = y,\quad x = 1$

$T:\ x = u + v,\quad y = v - u^2$

Solution:

$$x=\sqrt{-y}\Rightarrow u+v=\sqrt{-v+u^2}$$
$$u^2+2uv+v^2=-v+u^2$$
$$v^2+2uv+v^2=0$$
$$v(v+2u+1)=0$$
$$v=0 \quad \text{or} \quad v=-2u-1$$

$$x=1\Rightarrow u+v=1$$

$$x-y\Rightarrow u+v=v-u^2\rightarrow\Rightarrow u+u^2=0\Rightarrow u(u+1)=0\Rightarrow u=0 \text{ or } u=-1$$

$$\begin{cases} u=0 \\ v=0 \\ u+v=1 \end{cases} \qquad \begin{cases} u=-1 \\ v=-2u-1 \\ u+v=1 \end{cases}$$

$$(0,0):\quad \left.\begin{matrix} u+v=0 \\ v-u^2=0 \end{matrix}\right\}\Rightarrow -u-u^2=0\Rightarrow \begin{cases} u=0 \text{ or } -1 \\ v=0 \text{ or } -1 \end{cases}\Rightarrow (0,0),(-1,1)$$

$$(1,1):\quad \left.\begin{matrix} u+v=1 \\ v-u^2=1 \end{matrix}\right\}\Rightarrow u=0 \text{ or } -1\Rightarrow \begin{cases} (0,1) \\ (-1,2) \end{cases}$$

$$(1,-1):\quad \left.\begin{matrix} u+v=1 \\ v-u^2=-1 \end{matrix}\right\}\Rightarrow u^2+u-2=0\Rightarrow (u+2)(u-1)=0$$
$$\Rightarrow \begin{cases} u=-2 \text{ or } 1 \\ v=3 \text{ or } 0 \end{cases}\Rightarrow \begin{cases} (-2,3) \\ (1,0) \end{cases}$$

$$(0,0)\rightarrow(1,-1)\Rightarrow(0,0)\rightarrow(1,0) \quad \text{or} \quad (-1,1)\rightarrow(-2,3)$$
$$(1,-1)\rightarrow(1,1)\Rightarrow(1,0)\rightarrow(0,1) \quad \text{or} \quad (-2,3)\rightarrow(-1,2)$$

S_1: $+$ $\qquad$ S_2: $-$

Use S_1:

$$J=\begin{vmatrix} 1 & 1 \\ -2u & 1 \end{vmatrix}=1+2u$$

$$\int_0^1\int_0^{1-v}\frac{1}{\left[4\left(u+v\right)-4\left(v-u^2\right)+1\right]^2}\left(1+2u\right)dudv$$

$$=\int_0^1\int_0^{1-v}\frac{1}{\left[4u+4v-4v+4u^2+1\right]^2}\left(1+2u\right)dudv$$

$$=\int_0^1\int_0^{1-v}\frac{1}{\left[\left(1+2u\right)^2\right]^2}\left(1+2u\right)dudv=\int_0^1\int_0^{1-v}\left(1+2u\right)^{-3}dudv$$

$$=\frac{1}{2}\int_0^1\frac{\left(1+2u\right)^{-2}}{-2}\Bigg]_0^{1-v}dv$$

$$=-\frac{1}{4}\int_0^1\left[\frac{1}{\left[1+2\left(1-v\right)\right]^2}-\frac{1}{1}\right]dv$$

$$=\frac{1}{4}\int_0^1\left[1-\frac{1}{\left(3-2v\right)^2}\right]dv$$

$$=\frac{1}{4}\left[v-\frac{1}{2}\frac{1}{3-2v}\right]_0^1$$

$$=\frac{1}{4}\left[1-\frac{1}{2}\frac{1}{1}+\frac{1}{6}\right]=\frac{1}{4}\left(\frac{1}{2}+\frac{1}{6}\right)=\frac{1}{6}$$