MATH 267 CHAPTER 17 MULTIPLE INTEGRALS

17.8 SPHERICAL COORDINATES

The spherical coordinates of a point *P* are given by an ordered triple (ρ, ϕ, θ), where $\rho = \left\| \overrightarrow{OP} \right\|$ (O is the origin), ϕ is the angle between $\overrightarrow{OP}$ and k, and θ is the polar angle associated with the projection P' of *P* onto the *xy*-plane.

Note that $\rho \geq 0$ and $0 \leq \theta \leq \pi$.

Theorem The rectangular coordinates (x, y, z) and the **spherical coordinates** (ρ, ϕ, θ) of a point *P* are related as follows:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

Proof:

$$\begin{aligned} x &= \left\| \overrightarrow{OP'} \right\| \cos\theta & y &= \left\| \overrightarrow{OP'} \right\| \sin\theta & & \\ &= \left\| \overrightarrow{QP} \right\| \cos\theta & &= \left\| \overrightarrow{QP} \right\| \sin\theta & z &= \rho\cos\phi \\ &= \rho \sin\phi \cos\theta & &= \rho \sin\phi \sin\theta & & \end{aligned}$$

Spherical to Cartesian

2) $\left(1, \frac{3\pi}{4}, \frac{2\pi}{3}\right) \to (x, y, z)$

Solution:

$$x = \rho \sin\phi \cos\theta = (1)\sin\frac{3\pi}{4}\cos\frac{2\pi}{3} = 1\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) = -\frac{\sqrt{2}}{4}$$

$$y = \rho \sin\phi \sin\theta = (1)\sin\frac{3\pi}{4}\sin\frac{2\pi}{3} = 1\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}}{4}$$

$$z = \rho\cos\phi = (1)\cos\frac{3\pi}{4} = 1\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}$$

Thus, $\left(1, \frac{3\pi}{4}, \frac{2\pi}{3}\right) \to \left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{2}\right)$.

Cartesian to Spherical and then to Cylindrical

4) $\left(1, \sqrt{3}, 0\right) \to (\rho, \phi, \theta) \to (r, \theta, z)$

Solution:

$$\left.\begin{aligned} x &= 1 = \rho \sin\phi \cos\theta \\ y &= \sqrt{3} = \rho \sin\phi \sin\theta \end{aligned}\right\} \Rightarrow \frac{\sin\theta}{\cos\theta} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

$$z = 0 = \rho\cos\phi \Rightarrow \cos\phi = 0 \Rightarrow \phi = \frac{\pi}{2}$$

$$1 = \rho\cos\frac{\pi}{3} \Rightarrow 1 = \rho \cdot \frac{1}{2} \Rightarrow \rho = 2$$

Thus, $\left(1, \sqrt{3}, 0\right) \to \left(2, \frac{\pi}{2}, \frac{\pi}{3}\right)$.

$$x^2+y^2=r^2 \Rightarrow 1+3=r^2 \Rightarrow r=2$$

$$\tan\theta=\frac{y}{x}=\frac{\sqrt{3}}{1} \Rightarrow \theta=\frac{\pi}{3}$$

Thus, the cylindrical coordinates are $\left(2,\frac{\pi}{3},0\right)$.

Graphs: Spherical to Cartesian

6)(a) $\rho=5$ **(b)** $\phi=\frac{2\pi}{3}$ **(c)** $\theta=\frac{\pi}{4}$

Solution:

(a) sphere $x^2+y^2+z^2=25$

(b) half-cone with vertex angle $2\cdot\frac{2\pi}{3}=\frac{4\pi}{3}$

(c) plane with edge on the *z*-axis making an angle of $\frac{\pi}{4}$ with the *xz*-plane

8) $\rho\sec\phi=6$

Solution:

$$\rho\sec\phi=6 \Rightarrow \rho=6\cos\phi \Rightarrow \rho=6\cdot\frac{z}{\rho} \Rightarrow \rho^2=6z$$

$$x^2+y^2+z^2=6z \Rightarrow x^2+y^2+(z-3)^2=9 \text{ sphere } C(0,0,3),\ r=3$$

10) $\rho=4\sec\phi$

Solution: $\rho=4\sec\phi \Rightarrow \rho=\dfrac{4}{\cos\phi} \Rightarrow \rho\cos\phi=4 \Rightarrow z=4$ plane

12) $\rho=8\sin\phi\sin\theta$

Solution:

$$\rho=8\sin\phi\sin\theta \Rightarrow \rho^2=8\rho\sin\phi\sin\theta$$

$$x^2+y^2+z^2=8y$$

$$x^2+(y-4)^2+z^2=16 \text{ sphere } C(0,4,0),\ r=4$$

14) $\rho=2\csc\phi\sec\theta$

Solution:

$$\rho=2\csc\phi\sec\theta \Rightarrow \rho=\frac{2}{\sin\phi\cos\theta} \Rightarrow \rho\sin\phi\cos\theta=2$$

$x=2$ plane

16) $\rho\sin\phi=3$

Solution: $\rho\sin\phi=3 \Rightarrow \begin{cases}\rho\sin\phi\cos\theta=3\cos\theta \\ \rho\sin\phi\sin\theta=3\sin\theta\end{cases} \Rightarrow \begin{cases}x=3\cos\theta \\ y=3\sin\theta\end{cases} \Rightarrow x^2+y^2=9$

18) $\tan\theta = 4$
Solution:

$$\tan\theta = 4 \Rightarrow \theta = \tan^{-1}4 \quad \text{plane}$$

$$\frac{\sin\theta}{\cos\theta} = 4 \Rightarrow \sin\theta = 4\cos\theta \Rightarrow \rho\sin\phi\sin\theta = 4\rho\sin\phi\cos\theta$$

$$y = 4x$$

20) $\rho^2 - 3\rho + 2 = 0$
Solution:

$$\rho^2 - 3\rho + 2 = 0 \Rightarrow (\rho - 2)(\rho - 1) = 0 \Rightarrow \rho = 2 \text{ or } \rho = 1$$

spheres with $C(0,0,0)$ and radii 2 and 1

Graphs: Cartesian to Spherical
22) $x^2 + y^2 = 4z$
Solution:

$$x^2 + y^2 = 4z$$
$$x^2 + y^2 + z^2 = z^2 + 4z$$
$$x^2 + y^2 + z^2 = (z+2)^2 - 4$$
$$\rho = (\rho\cos\phi + 2)^2 - 4$$
$$\rho = \rho^2\cos^2\phi + 4\rho\cos\phi$$
$$1 = \rho\cos^2\phi + 4\cos\phi$$

or

$$(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = 4\rho\cos\phi$$
$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = 4\rho\cos\phi$$
$$\rho^2\sin^2\phi = 4\rho\cos\phi$$
$$\rho^2\sin^2\phi - 4\rho\cos\phi = 0$$
$$\rho(\rho\sin^2\phi - 4\cos\phi) = 0$$
$$\rho = 0 \quad \text{or} \quad \rho\sin^2\phi = 4\cos\phi$$

24) $y = x$

Solution: $\rho\sin\phi\sin\theta = \rho\sin\phi\cos\theta \Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \dfrac{\pi}{4}$

26) $x^2 + (y-2)^2 = 4$
Solution:

$$\rho^2\sin^2\phi\cos^2\theta + (\rho\sin\phi\sin\theta - 2)^2 = 4$$
$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta - 4\rho\sin\phi\sin\theta = 0$$
$$\rho^2\sin^2\phi - 4\rho\sin\phi\sin\theta = 0$$
$$\rho = 0 \quad \text{or} \quad \rho\sin^2\phi = 4\sin\phi\sin\theta$$
$$\rho\sin\phi = 4\sin\theta$$
$$\rho = 4\sin\theta\csc\phi$$

28) $x^2 - y^2 - z^2 = 1$

Solution: $\rho^2\sin^2\phi\cos^2\theta - \rho^2\sin^2\phi\sin^2\theta - \rho^2\cos^2\phi = 1 \Rightarrow \rho^2 = \dfrac{1}{\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta - \cos^2\phi}$

Evaluation Theo rem $\iiint\limits_Q f(\rho,\phi,\theta)\,dV = \int_m^n\int_c^d\int_a^b f(\rho,\phi,\theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

32) Find the volume and the centroid of the solid bounded by the cone $z=\sqrt{x^2+y^2}$, the cylinder $x^2+y^2=4$, and the *xy*-plane.

Solution:

$$z^2 = x^2+y^2$$
$$\rho^2\cos^2\phi = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$$
$$\rho^2\cos^2\phi = \rho^2\sin^2\phi$$
$$\cos\phi = \sin\phi \Rightarrow \phi = \frac{\pi}{4}$$

$$x^2+y^2=4$$
$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = 4$$
$$\rho^2\sin^2\phi = 4$$
$$\rho^2 = 4\csc^2\phi \Rightarrow \rho = 2\csc\phi$$

$$V = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\csc\phi} \rho^2\sin\phi\, d\rho d\phi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left.\frac{\rho^3}{3}\sin\phi\right]_0^{2\csc\phi} d\phi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{8}{3}\csc^3\phi\sin\phi\, d\phi d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{8}{3}\csc^2\phi\, d\phi d\theta \; = \int_0^{2\pi} \left. -\frac{8}{3}\cot\phi\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta = \int_0^{2\pi}\left[-\frac{8}{3}\cdot 0 + \frac{8}{3}\cdot 1\right] d\theta = \frac{16\pi}{3}$$

$$M_{xy} = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\csc\phi} (\rho\cos\phi)\rho^2\sin\phi\, d\rho d\phi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left.\frac{\rho^4}{4}\right]_0^{2\csc\phi} \cos\phi\sin\phi\, d\phi d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{16}{4}\csc^4\phi\cos\phi\sin\phi\, d\phi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\csc^3\phi\cos\phi\, d\phi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4(\sin\phi)^{-3}\cos\phi\, d\phi d\theta$$

$$= 4\int_0^{2\pi} \left.\frac{(\sin\theta)^{-2}}{-2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta = -2\int_0^{2\pi}\left(\csc^2\frac{\pi}{2} - \csc^2\frac{\pi}{4}\right) d\theta = \ldots = 4\pi$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{4\pi}{16\pi/3} = \frac{3}{4} \text{ and the centroid is } \left(0,0,\frac{3}{4}\right)$$

34) Find the moment of inertia wrt a diameter of the base of a homogeneous solid hemisphere of radius *a*.

Solution: $z=\sqrt{a^2-x^2-y^2} \Rightarrow \rho = a$

$$I_x = \delta\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a \left(\rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi\right)\rho^2\sin\phi\, d\rho d\phi d\theta = \delta\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a \left(\rho^4\sin^3\phi\sin^2\theta + \rho^4\sin\phi\cos^2\phi\right) d\rho d\phi d\theta$$

$$= \delta\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(\sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi\right)\left.\frac{\rho^5}{5}\right]_0^a d\phi d\theta = \frac{\delta a^5}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left(\sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi\right) d\phi d\theta \text{...easier to do } \theta \text{ first...so}$$

$$I_x = \frac{\delta a^5}{5}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left(\sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi\right) d\theta d\phi = \frac{\delta\pi a^5}{5}\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}\sin^3\phi\left(\theta - \frac{1}{2}\sin 2\theta\right) + \left(\sin\phi\cos^2\phi\right)\theta\right]_0^{2\pi} d\phi$$

$$= \frac{\delta a^5}{5}\int_0^{\frac{\pi}{2}}\left(\pi\sin^3\phi + 2\pi\sin\phi\cos^2\phi\right) d\phi = \frac{\delta\pi a^5}{5}\int_0^{\frac{\pi}{2}}\left(\left(1-\cos^2\phi\right)\sin\phi + 2\sin\phi\cos^2\phi\right) d\phi$$

$$= \frac{\delta\pi a^5}{5}\left[-\cos\phi + \frac{\cos^3\phi}{3} - \frac{2\cos^3\phi}{3}\right]_0^{\frac{\pi}{2}} = \frac{\delta\pi a^5}{5}\left(1-1+\frac{2}{3}\right) = \frac{2\delta\pi a^5}{15}$$