MATH 267 CHAPTER 17 MULTIPLE INTEGRALS

17.7 CYLINDRICAL COORDINATES

2-D Polar (r,θ)

3-D Cylindrical (r,θ,z)

Theorem The rectangular coordinates (x,y,z) and the **cylindrical coordinates** (r,θ,z) of a point P are related as follows:

$$x = r\cos\theta,\quad y = r\sin\theta,\quad \tan\theta = \frac{y}{x},$$
$$r^2 = x^2 + y^2,\qquad z = z$$

Cylindrical to Cartesian

2a) $r = -3$ **b)** $\theta = \frac{\pi}{4}$ **c)** $z = -2$

Solution:

a) $r = -3 \Rightarrow x^2 + y^2 = 9$ right circular cylinder of radius 3 with axis along the z-axis

b) $\theta = \frac{\pi}{4} \Rightarrow \tan\theta = 1 \Rightarrow \frac{y}{x} = 1 \Rightarrow y = x$ plane containing the z-axis and bisecting the first octant

c) $z = -2$ plane parallel to the xy-plane with z-intercept -2

4) $r = -\csc\theta$

Solution:

$r = -\csc\theta \Rightarrow r = -\frac{1}{\sin\theta} \Rightarrow r\sin\theta = -1 \Rightarrow y = -1$ plane parallel to the xz-plane with y-intercept -1

6) $z = 4 - r^2$

Solution:

$z = 4 - r^2 \Rightarrow z = 4 - x^2 - y^2$ paraboloid with axis on the z-axis, vertex $(0,0,4)$

8) $r\sec\theta = 4$

Solution:

$r\sec\theta = 4 \Rightarrow \frac{r}{\cos\theta} = 4 \Rightarrow r = 4\cos\theta \Rightarrow r^2 = 4r\cos\theta \Rightarrow x^2 + y^2 = 4x \Rightarrow (x+2)^2 + y^2 = 4$

right circular cylinder with trace $(x+2)^2 + y^2 = 4$ on the xy-plane

10) $3z = r$

Solution: $3z = r \Rightarrow 3z = \sqrt{x^2 + y^2} \Rightarrow 9z^2 = x^2 + y^2 \Rightarrow x^2 + y^2 - 9z^2 = 0$ cone

12) $r^2 + z^2 = 16$

Solution: $r^2 + z^2 = 16 \Rightarrow x^2 + y^2 + z^2 = 16$ sphere $C(0,0,0)$, $r = 4$

14) $r = \tan\theta\sec\theta$

Solution:

$$r = \tan\theta\sec\theta \Rightarrow r = \frac{\tan\theta}{\cos\theta} \Rightarrow r\cos\theta = \tan\theta \Rightarrow x = \frac{y}{x}$$

$\Rightarrow x^2 = y$ parabolic cylinder with rulings // z-axis

Cartesian to Cylindrical

Example 1 p. 934

$$z^2 = x^2 + y^2 \Rightarrow z^2 = r^2 \Rightarrow z = \pm r \text{ or } z = r \text{ circular cone on } z\text{-axis}$$

Exercises

16) $x^2 + y^2 = 4z$

Solution: $r^2 = 4z \Rightarrow z = \frac{1}{4}r^2$

18) $y = x$

Solution: $r\sin\theta = r\cos\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ or $\frac{\pi}{4} + n\pi$

20) $x^2 + (y-2)^2 = 4$

Solution: $x^2 + y^2 - 4y + 4 = 4 \Rightarrow r^2 - 4r\sin\theta = 0 \Rightarrow$ ~~$r = 0$~~ or $r = 4\sin\theta$

22) $x^2 - y^2 - z^2 = 1$

Solution: $r^2\cos^2\theta - r^2\sin^2\theta - z^2 = 1 \Rightarrow r^2\left(\cos^2\theta - \sin^2\theta\right) - z^2 = 1 \Rightarrow r^2\cos 2\theta - z^2 = 1$

Evaluation Theorem $\iiint_Q f(r,\theta,z)\,dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta$

26) see figure on p. 941

Solution: $4\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{4-r} f(r,\theta,z)\ r\,dz\,dr\,d\theta, \ r \ge 0$

28) see figure on p. 941

Solution: $\sqrt{32 - r^2} = r \Rightarrow 32 - r^2 = r^2 \Rightarrow 2r^2 = 32 \Rightarrow r = \pm 4$

$\int_0^{2\pi} \int_0^4 \int_r^{\sqrt{32-r^2}} f(r,\theta,z)\ r\,dz\,dr\,d\theta, \ r \ge 0$

30) A solid is bounded by the cone $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 4$, and the xy-plane. Find **(a)** its volume **(b)** its centroid.

Solution: $z = \sqrt{x^2 + y^2} \Rightarrow z = r$; $x^2 + y^2 = 4 \Rightarrow r = 2$

(a)

$$V = \int_0^{2\pi} \int_0^2 \int_0^r r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 rz\Big]_0^r\, dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 r^2\, dr\,d\theta$$
$$= \int_0^{2\pi} \frac{r^3}{3}\Big]_0^2\, d\theta$$
$$= \int_0^{2\pi} \frac{8}{3}\, d\theta = \frac{8}{3}\cdot 2\pi = \frac{16\pi}{3}$$

(b)

$$M_{xy} = \int_0^{2\pi} \int_0^2 \int_0^r z\,r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 r\,\frac{z^2}{2}\Big]_0^r\, dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 \frac{r^3}{2}\, dr\,d\theta$$
$$= \int_0^{2\pi} \frac{r^4}{8}\Big]_0^2\, d\theta = \int_0^{2\pi} 2\,d\theta = 4\pi$$

$$\bar{z} = \frac{M_{xy}}{V} = \frac{4\pi}{16\pi/3} = \frac{12}{16} = \frac{3}{4},\quad \bar{x} = \bar{y} = 0 \Rightarrow \text{centroid is } \left(0, 0, \frac{3}{4}\right)$$

32) A homogeneous solid of density δ is bounded by the cone $z=\sqrt{x^2+y^2}$ and the paraboloid $z=x^2+y^2$. Find its

(a) center of mass

(b) its moment of inertia wrt the *z*-axis

Solution:

(a) $M=\delta\int_0^{2\pi}\int_0^1\int_{r^2}^{r} rdzdrd\theta$

$$=\delta\int_0^{2\pi}\int_0^1 r\,z\Big]_{r^2}^{r} drd\theta=\delta\int_0^{2\pi}\int_0^1\left(r^2-r^3\right)drd\theta=\delta\int_0^{2\pi}\left[\frac{r^3}{3}-\frac{r^4}{4}\right]_0^1 d\theta=\delta\int_0^{2\pi}\left(\frac{1}{3}-\frac{1}{4}\right)d\theta=\ldots=\delta\frac{\pi}{6}$$

$$M_{xy}=\delta\int_0^{2\pi}\int_0^1\int_{r^2}^{r} zrdzdrd\theta=\delta\int_0^{2\pi}\int_0^1 r\frac{z^2}{2}\Big]_{r^2}^{r} drd\theta=\delta\int_0^{2\pi}\int_0^1\left(\frac{r^3}{2}-\frac{r^5}{2}\right)drd\theta=\delta\int_0^{2\pi}\left[\frac{r^4}{8}-\frac{r^6}{12}\right]_0^1 d\theta$$

$$=\delta\int_0^{2\pi}\left(\frac{1}{8}-\frac{1}{12}\right)d\theta=\ldots\frac{1}{12}\delta\pi$$

$$\bar{x}=0,\quad \bar{y}=0,\quad \bar{z}=\frac{M_{xy}}{M}=\frac{\frac{1}{12}\delta\pi}{\frac{1}{6}\delta\pi}=\frac{1}{2}\Rightarrow\text{center of mass }\left(0,0,\frac{1}{2}\right)$$

(b) $I_z=\delta\int_0^{2\pi}\int_0^1\int_{r^2}^{r} r^2\, rdzdrd\theta=\delta\int_0^{2\pi}\int_0^1 r^3 z\Big]_{r^2}^{r} drd\theta=\delta\int_0^{2\pi}\int_0^1\left(r^4-r^5\right)drd\theta=\ldots=\frac{1}{15}\delta\pi$

34) A solid is bounded by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=4$, and the density at $P(x,y,z)$ is directly proportional to the distance from the *z*-axis to *P*. Find its mass.

Solution:

$$M=\int_0^{2\pi}\int_0^4\int_r^4 (kr)\,rdzdrd\theta=\int_0^{2\pi}\int_0^4 kr^2z\Big]_r^4 drd\theta=\int_0^{2\pi}\int_0^4\left(4kr^2-kr^3\right)drd\theta$$

$$=\int_0^{2\pi}\left[4k\frac{r^3}{3}-k\frac{r^4}{4}\right]_0^4 d\theta=\ldots\frac{128}{3}\pi k$$

36) Find the moment of inertia wrt the *z*-axis for the solid described in Exercise 34.

Solution: $I_z=\int_0^{2\pi}\int_0^4\int_r^4 r^2(kr)\,rdzdrd\theta=\int_0^{2\pi}\int_0^4 kr^4z\Big]_r^4 drd\theta=\ldots=\frac{4^6k}{15}\pi$