MATH 267 CHAPTER 17 MULTIPLE INTEGRALS

17.6 MOMENTS AND CENTER OF MASS

Definition Let L be a lamina that has the shape of a region R in the xy – plane. If the area mass density at (x,y) is $\delta(x,y)$ and if δ is continuous on R, then the **mass** m, the **moments** M_x and M_y, and the **center of mass** $(\bar{x},\bar{y})$ are given by

(i) $m = \iint_R \delta(x,y)\,dA$

(ii) $M_x = \iint_R y\delta(x,y)\,dA, \quad M_y = \iint_R x\delta(x,y)\,dA$

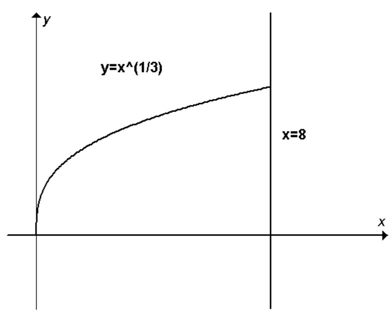
(iii) $\bar{x} = \dfrac{M_y}{m} = \dfrac{\iint_R x\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}, \qquad \bar{y} = \dfrac{M_x}{m} = \dfrac{\iint_R y\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}$

If L is homogeneous then the area mass density $\delta(x,y)$ is a constant and can be cancelled. Thus, the center of mass depends only on the shape of the homogeneous lamina and we call $(\bar{x},\bar{y})$ the **centroid** of the region R.

Exercises

2) $y = \sqrt[3]{x}, \ x = 8, \ y = 0, \ \delta(x,y) = y^2$

Solution:



$$m = \int_0^8 \int_0^{\sqrt[3]{x}} y^2\,dy\,dx = \int_0^8 \frac{y^3}{3}\Bigg]_0^{\sqrt[3]{x}} dx = \int_0^8 \frac{x}{3}dx = \frac{1}{3}\frac{x^2}{2}\Bigg]_0^8 = \frac{32}{3}$$

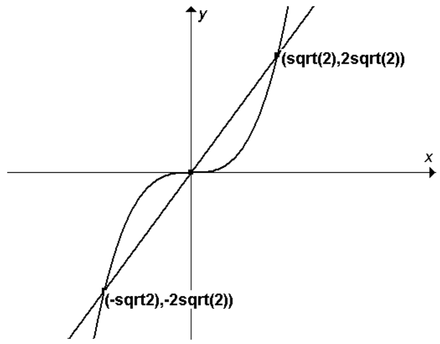
$$M_x = \int_0^8 \int_0^{\sqrt[3]{x}} y^3\,dy\,dx = \int_0^8 \frac{y^4}{4}\Bigg]_0^{\sqrt[3]{x}} dx = \int_0^8 \frac{x^{4/3}}{4}dx = \frac{1}{4}x^{7/3}\cdot\frac{3}{7}\Bigg]_0^8 = \frac{3}{28}8^{7/3} = 3\cdot\frac{32}{7}$$

$$M_y = \int_0^8 \int_0^{\sqrt[3]{x}} xy^2\,dy\,dx = \int_0^8 x\frac{y^3}{3}\Bigg]_0^{\sqrt[3]{x}} dx = \int_0^8 \frac{x^2}{3}dx = \frac{x^3}{9}\Bigg]_0^8 = \frac{8^3}{9}$$

$$\bar{x} = \frac{M_y}{m} = \frac{16}{3}; \quad \bar{y} = \frac{M_x}{m} = \frac{9}{7} \Rightarrow (\bar{x},\bar{y}) = \left(\frac{16}{3},\frac{9}{7}\right)$$

4) $y = x^3$, $y = 2x$; density at $P(x,y)$ is directly proportional to the distance from the x-axis to P

Solution:



$$\delta(x,y) = k|y|$$

$$x^3 = 2x \Rightarrow x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

$$m = 2\int_0^{\sqrt{2}}\int_{x^3}^{2x} ky\,dy\,dx = \int_0^{\sqrt{2}} k\left.\frac{y^2}{2}\right]_{x^3}^{2x} dx = \frac{2k}{2}\int_0^{\sqrt{2}}\left[(2x)^2 - (x^3)^2\right]dx$$

$$= k\int_0^{\sqrt{2}}\left[4x^2 - x^6\right]dx = k\left[\frac{4x^3}{3} - \frac{x^7}{7}\right]_0^{\sqrt{2}} = k\left[\frac{4(\sqrt{2})^3}{3} - \frac{(\sqrt{2})^7}{7}\right] = \frac{32}{21}k\sqrt{2}$$

$$M_x = \int_{-\sqrt{2}}^{0}\int_{2x}^{x^3} -ky^2\,dy\,dx + \int_0^{\sqrt{2}}\int_{x^3}^{2x} ky^2\,dy\,dx$$

$$= \int_{-\sqrt{2}}^{0} -\left.\frac{ky^3}{3}\right]_{2x}^{x^3} dx + \int_0^{\sqrt{2}}\left.\frac{ky^3}{3}\right]_{x^3}^{2x} dx = -\frac{k}{3}\int_{-\sqrt{2}}^{0}\left(x^9 - 8x^3\right)dx + \frac{k}{3}\int_0^{\sqrt{2}}\left(8x^3 - x^9\right)dx$$

$$= -\frac{k}{3}\left[\frac{x^{10}}{10} - \frac{8x^4}{4}\right]_{-\sqrt{2}}^{0} + \frac{k}{3}\left[\frac{8x^4}{4} - \frac{x^{10}}{10}\right]_0^{\sqrt{2}} = -\frac{k}{3}\left[-\frac{2^5}{10} + 2\cdot 2^2\right] + \frac{k}{3}\left[2\cdot 2^2 - \frac{2^5}{10}\right] = 0$$

$$M_y = \int_{-\sqrt{2}}^{0}\int_{2x}^{x^3} x(-ky)\,dy\,dx + \int_0^{\sqrt{2}}\int_{x^3}^{2x} x(ky)\,dy\,dx$$

$$= \int_{-\sqrt{2}}^{0} -\left.\frac{kxy^2}{2}\right]_{2x}^{x^3} dx + \int_0^{\sqrt{2}}\left.\frac{kxy^2}{2}\right]_{x^3}^{2x} dx = -\frac{k}{2}\int_{-\sqrt{2}}^{0}\left(x^7 - 4x^3\right)dx + \frac{k}{2}\int_0^{\sqrt{2}}\left(4x^3 - x^7\right)dx$$

$$= -\frac{k}{2}\int_{-\sqrt{2}}^{\sqrt{2}}\left(4x^3 - x^7\right)dx = 0$$

Thus, $\bar{x} = \dfrac{M_y}{m} = 0$, $\quad \bar{y} = \dfrac{M_x}{m} = 0$.

Moments and center of mass in three dimensions

(i) $m = \iiint_Q \delta(x,y,z)\,dV$

(ii) $M_{xy} = \iiint_Q z\delta(x,y,z)\,dV$

$M_{xz} = \iiint_Q y\delta(x,y,z)\,dV$

$M_{yz} = \iiint_Q x\delta(x,y,z)\,dV$

(iii) $\bar{x} = \dfrac{M_{yz}}{m}$; $\bar{y} = \dfrac{M_{xz}}{m}$; $\bar{z} = \dfrac{M_{xy}}{m}$

Exercise

18) Q is the tetrahedron bounded by the coordinate planes and the plane $2x+5y+z=10$. The density at $P(x,y,z)$ is directly proportional to the distance from the xz-plane to P.
Solution:

$$\delta(x,y,z)=k|y|=ky$$

$$m=\int_0^5\int_0^{\frac{10-2x}{5}}\int_0^{10-2x-5y} kydzdydx=\frac{25}{3}k$$

$$M_{yz}=\int_0^5\int_0^{\frac{10-2x}{5}}\int_0^{10-2x-5y} x(ky)dzdydx=\frac{25}{3}k$$

$$M_{xz}=\int_0^5\int_0^{\frac{10-2x}{5}}\int_0^{10-2x-5y} y(ky)dzdydx=\frac{20}{3}k$$

$$M_{xy}=\int_0^5\int_0^{\frac{10-2x}{5}}\int_0^{10-2x-5y} z(ky)dzdydx=\frac{50}{3}k$$

Thus, $\bar{x}=\frac{M_{yz}}{m}=1,\ \bar{y}=\frac{M_{xz}}{m}=\frac{4}{5},\ \bar{z}=\frac{M_{xy}}{m}=2$.

Remarks M_x and M_y are also called the **<u>first moments</u>** of L with respect to the coordinate axes. If we use the squares of the distances from the coordinate axes, we obtain the **<u>second moments</u>**, or **<u>moments of inertia</u>**, I_x and I_Y with respect to the x-axis and y-axis, resp. The sum $I_o=I_x+I_y$ is the **<u>polar moment of inertia</u>**, or the **<u>moment of inertia with respect to the origin</u>**.

Definition Moments of inertia of a lamina

$$I_x=\lim_{\|P\|\to 0}\sum_k y_k^2\,\delta(x_k,y_k)\Delta A_k=\iint_R y^2\delta(x,y)dA$$

$$I_y=\lim_{\|P\|\to 0}\sum_k x_k^2\,\delta(x_k,y_k)\Delta A_k=\iint_R x^2\delta(x,y)dA$$

$$I_o=\lim_{\|P\|\to 0}\sum_k\left(x_k^2+y_k^2\right)\delta(x_k,y_k)\Delta A_k=\iint_R\left(x^2+y^2\right)\delta(x,y)dA$$

Remarks Moments of inertia are useful in problems that involve rotation of an object about a fixed axis, such as rotation of a wheel (or disk) about an axle. If a particle P on a wheel has mass m and is a distance k from the axis of rotation, then the moment of inertia I of P with respect to the axis is mk^2. If the angular speed $d\theta/dt$ is a constant ω, then the speed v of the particle is $k\omega$. From physics, the **<u>kinetic energy</u>** $K.E.$ of P is

$$K.E.=\tfrac{1}{2}mv^2.$$

Since $v=k\omega$,

$$K.E.=\tfrac{1}{2}mk^2\omega^2=\tfrac{1}{2}I\omega^2.$$

Thus, the kinetic energy is directly proportional to the moment of inertia. For a fixed ω, the larger the moment of inertia, the larger the amount of work required to stop the rotation.

Exercises

10) Find I_x, I_y, I_0 for the lamina in Exercise 2: $y = \sqrt[3]{x},\ \ x = 8,\ \ y = 0,\ \ \delta(x,y) = y^2$

Solution:

$$I_x = \int_0^8 \int_0^{\sqrt[3]{x}} y^2 y^2 dydx = \frac{96}{5}$$

$$I_y = \int_0^8 \int_0^{\sqrt[3]{x}} x^2 y^2 dydx = \frac{1024}{3}$$

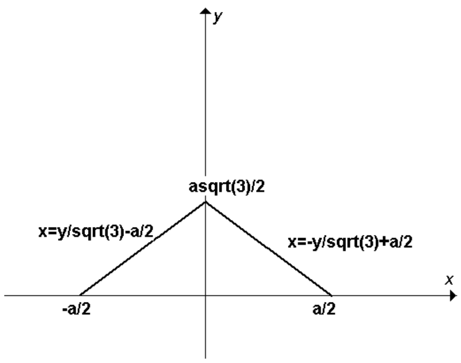
$$I_0 = \frac{96}{5} + \frac{1024}{3} = \frac{5408}{15}$$

14) A homogeneous lamina has the shape of an equilateral triangle of side *a*. Find the moment of inertia wrt

a) an altitude **b)** a side **c)** a vertex

Solution:

a)



$$I_y = 2\delta \int_0^{\sqrt{3}a/2} \int_0^{-y/\sqrt{3}+a/2} x^2 dxdy$$

$$= \frac{2}{3}\delta \int_0^{\sqrt{3}a/2} \left(-\frac{y}{\sqrt{3}} + \frac{a}{2}\right)^3 dy$$

$$= \frac{2}{3}\delta \int_0^{\sqrt{3}a/2} \left(-\frac{y^3}{3\sqrt{3}} + \frac{ay^2}{2} - \frac{3a^2 y}{4\sqrt{3}} + \frac{a^3}{8}\right) dy$$

$$= \frac{2}{3}\delta \left[-\frac{9a^4}{(16)(4)(3\sqrt{3})} + \frac{3\sqrt{3}a^4}{(8)(3)(2)} - \frac{(3)(3)a^4}{(4)(2)(4\sqrt{3})} + \frac{\sqrt{3}a^4}{(2)(8)}\right] = \frac{\sqrt{3}a^4\delta}{96}$$

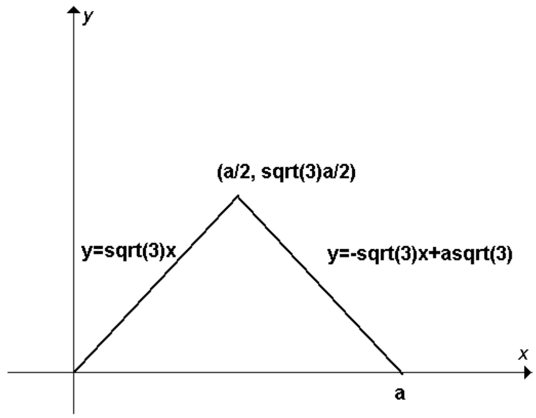
b) See Figure in (a)

$$I_x = 2\delta \int_0^{\sqrt{3}a/2} \int_0^{-y/\sqrt{3}+a/2} y^2 dxdy$$

$$= 2\delta \int_0^{\sqrt{3}a/2} \left(-\frac{y^3}{\sqrt{3}} + \frac{ay^2}{2}\right) dy$$

$$= \delta \left[\frac{3\sqrt{3}a^4}{24} - \frac{9a^4}{32\sqrt{3}}\right] = \frac{\sqrt{3}a^4\delta}{32}$$

c)



$$I_0 = \delta\int_0^{\sqrt{3}a/2}\int_{y/\sqrt{3}}^{(\sqrt{3}a-y)/\sqrt{3}}\left(x^2+y^2\right)dxdy$$

$$= \delta\int_0^{\sqrt{3}a/2}\left[\frac{\left[a-y/\sqrt{3}\right]^3}{3}-\frac{y^3}{9\sqrt{3}}+y^2\left(a-\frac{y}{\sqrt{3}}-\frac{y}{\sqrt{3}}\right)\right]dy$$

$$= \delta\left[\frac{-\sqrt{3}\left[a-y/\sqrt{3}\right]^4}{12}-\frac{y^4}{36\sqrt{3}}+\frac{ay^3}{3}-\frac{y^4}{2\sqrt{3}}\right]_0^{\sqrt{3}a/2} = \ldots = \frac{5\sqrt{3}a^4\delta}{48}$$

Moments of Inertia of solids

$$I_z = \iiint_Q \left(x^2+y^2\right)\delta(x,y,z)dV$$

$$I_x = \iiint_Q \left(y^2+z^2\right)\delta(x,y,z)dV$$

$$I_y = \iiint_Q \left(x^2+z^2\right)\delta(x,y,z)dV$$

Exercise Set up an iterated integral that can be used to find the moment of inertia wrt the *z*-axis of the indicated solid.

26) *Q:* cone bounded by the graphs of $x^2+9y^2-z^2=0$ and $z=36$; $\delta(x,y,z)=x^2+y^2$

Solution:

$$I_z = \int_0^{12}\int_0^{\sqrt{36^2-9y^2}}\int_{\sqrt{x^2+9y^2}}^{36}\left(x^2+y^2\right)^2 dzdxdy$$

or $$I_z = \int_0^{36}\int_{-z/3}^{z/3}\int_{-\sqrt{z^2-9y^2}}^{\sqrt{z^2-9y^2}}\left(x^2+y^2\right)^2 dxdydz$$

Theorem of Pappus Let *R* be a region in a plane that lies entirely on one side of a line *l* in the plane. If *R* is revolved once about *l*, the volume of the resulting solid is the product of the area of *R* and the distance traveled by the centroid of *R*.