MATH 267 CHAPTER 17 MULTIPLE INTEGRALS

17.5 TRIPLE INTEGRALS

Definition $\iiint_Q f(x,y,z)dV = \lim_{\|P\|\to 0} \sum_k f(u_k,v_k,w_k)\Delta V_k$

Evaluation Theorem $\iiint_Q f(x,y,z)dV = \iint_R \left[\int_{k_1(x,y)}^{k_2(x,y)} f(x,y,z)dz\right]dA$

Exercises Evaluate the iterated integral.

2) $\int_0^1\int_{-1}^2\int_1^3 \left(6x^2z+5xy^2\right)dzdxdy$

Solution:

$$\int_0^1\int_{-1}^2\int_1^3 \left(6x^2z+5xy^2\right)dzdxdy = \int_0^1\int_{-1}^2\left[3x^2z^2+5xy^2z\right]_1^3\ dxdy$$
$$= \int_0^1\int_{-1}^2\left[27x^2+15xy^2-3x^2-5xy^2\right]dxdy = \int_0^1\int_{-1}^2\left[24x^2+10xy^2\right]dxdy$$
$$= \int_0^1\left[8x^3+5x^2y^2\right]_{-1}^2\ dy = \int_0^1\left(64+20y^2+8-5y^2\right)dy = \left[72y+5y^3\right]_0^1 = 77$$

6) $\int_2^3\int_0^{3y}\int_1^{yz}\left(2x+y+z\right)dxdzdy$

Solution:

$$\int_2^3\int_0^{3y}\int_1^{yz}\left(2x+y+z\right)dxdzdy = \int_2^3\int_0^{3y}\left[x^2+xy+xz\right]_1^{yz}\ dzdy$$
$$= \int_2^3\int_0^{3y}\left[y^2z^2+y^2z+yz^2-1-y-z\right]dzdy = \int_2^3\left[y^2\frac{z^3}{3}+y^2\frac{z^2}{2}+y\frac{z^3}{3}-z-yz-\frac{z^2}{2}\right]_0^{3y}\ dy$$
$$= \int_2^3\left[\frac{27}{3}y^5+\frac{9}{2}y^4+\frac{27}{3}y^4-3y-3y^2-\frac{9}{2}y^2\right]dy = \left[9\frac{y^6}{6}+\frac{9}{2}\frac{y^5}{5}+9\frac{y^5}{5}-3\frac{y^2}{2}-y^3-\frac{9}{2}\frac{y^3}{3}\right]_0^3 = \ldots$$

If f is an arbitrary continuous function of three variables and Q is the region shown in the figure, express $\iiint_Q f(x,y,z)dV$ as an iterated triple integral in six different ways.

8) see figure on p. 924

Solution:

x first: $\int_0^2\int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)dxdydz$

$\int_{-3}^3\int_0^2\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)dxdzdy$

y first: $\int_0^2\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y,z)dydxdz$

$\int_{-3}^3\int_0^2\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y,z)dydzdx$

z first: $\int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^2 f(x,y,z)dzdxdy$

$\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^2 f(x,y,z)dzdydx$

Sketch the region bounded by the graphs of the equations, and use a triple integral to find its volume.

12) $x^2+z^2=4,\ \ y^2+z^2=4$

Solution:

$$V=8\int_0^2\int_0^{\sqrt{4-z^2}}\int_0^{\sqrt{4-z^2}}dydxdz=8\int_0^2\int_0^{\sqrt{4-z^2}}\sqrt{4-z^2}\ dxdz$$
$$=8\int_0^2\left(4-z^2\right)dz=8\left[4z-\frac{z^3}{3}\right]_0^2=\frac{128}{3}$$

14) $z=4y^2,\ z=2,\ x=2,\ \ x=0$

Solution:

$$V=2\int_0^2\int_0^{1/\sqrt{2}}\int_{4y^2}^2 dzdydx=2\int_0^2\int_0^{1/\sqrt{2}}\left(2-4y^2\right)dydx$$
$$=2\int_0^2\left[2y-4\frac{y^3}{3}\right]_0^{1/\sqrt{2}}dx=2\int_0^2\left[\frac{2}{\sqrt{2}}-\frac{4}{3}\cdot\frac{1}{2\sqrt{2}}\right]dx=2\int_0^2\frac{12-4}{6\sqrt{2}}dx=\ldots\frac{8}{3}\sqrt{2}$$

16) $z=x^2+y^2,\ \ y+z=2$

Solution:

$$V=2\int_{-2}^1\int_{y^2}^{2-y}\int_0^{\sqrt{z-y^2}}dxdzdy=2\int_{-2}^1\int_{y^2}^{2-y}\sqrt{z-y^2}dzdy$$
$$=2\int_{-2}^1\left(z-y^2\right)^{3/2}\cdot\frac{2}{3}\Bigg]_{y^2}^{2-y}dy=\frac{4}{3}\int_{-2}^1\left(2-y-y^2\right)^{3/2}dy-\frac{4}{3}\int_{-2}^1\left(2-2y^2\right)^{3/2}dy$$

18) $z=e^{x+y},\ \ y=3x,\ \ x=2,\ \ y=0,\ \ z=0$

Solution:

$$V=\int_0^2\int_0^{3x}\int_0^{e^{x+y}}dzdydx=\int_0^2\int_0^{3x}e^{x+y}dydx$$
$$=\int_0^2 e^{x+y}\Big]_0^{3x}dx=\int_0^2\left(e^{4x}-e^x\right)dx$$
$$=\left[\frac{1}{4}e^{4x}-e^x\right]_0^2=\frac{1}{4}e^8-e^2-\frac{1}{4}+1\approx 738.6$$

20) $y=x^2+z^2,\ \ z=x^2,\ \ z=4,\ \ \ y=0$

Solution:

$$V=2\int_0^2\int_{x^2}^4\int_0^{x^2+z^2}dydzdx=2\int_0^2\int_{x^2}^4\left(x^2+z^2\right)dzdx$$
$$=2\int_0^2\left[x^2z+\frac{z^3}{3}\right]_{x^2}^4dx=\int_0^2\left(4x^2+\frac{64}{3}-x^4-\frac{x^6}{3}\right)dx=\ldots\approx 81.68$$

Describe Q.

24) $\int_0^1\int_{z^3}^{\sqrt{z}}\int_0^{4-x}dydxdz$

Solution:

$0\le y\le 4-x,\ \ \ z^3\le x\le\sqrt{z},\ \ \ \ 0\le z\le 1$

Q is the region bounded by the planes $y=0$ and $y=4-x$ and the cylinders $x=z^3$ and $x=\sqrt{z}$.

26) $\int_0^1 \int_x^{3x} \int_0^{xy} dzdydx$
Solution:
$0 \le z \le xy, \quad x \le y \le 3x, \quad 0 \le x \le 1$
Q is the region under the surface $z = xy$ and over the triangular region bounded by
$y = x,\ y = 3x,$ and $x = 1$.

To find the mass of a solid

Definition of mass density $\delta(x,y,z) = \lim_{\| \Delta V_k \| \to 0} \frac{\Delta m_k}{\Delta V_k}$

$$\Delta m_k \approx \delta(x,y,z)\Delta V_k \text{ if limit exists and } \|\Delta V_k\| \approx 0$$

Mass of a solid $m = \iiint\limits_Q \delta(x,y,z)\,dV$

Mass of a lamina

$$\delta(x,y) = \lim_{\|A_k\| \to 0} \frac{\Delta m_k}{\Delta A_k}$$

$$m = \iint\limits_R \delta(x,y)\,dA$$

Exercises Set up an iterated double integral that can be used to find the mass of the lamina.

30) $\delta(x,y) = x^2 + y^2;\ \ xy^2 = 1,\ \ x = 0,\ \ y = 1,\ \ y = 2$
Solution:

$$m = \iint\limits_R \left(x^2 + y^2\right) dA = \int_1^2 \int_0^{1/y^2} \left(x^2 + y^2\right) dxdy$$

Set up an iterated triple integral that can be used to find the mass of the solid.

32) $\delta(x,y,z) = z + 1;\ \ z = 4 - x^2 - y^2,\ \ z = 0$
Solution:

$$m = \iiint\limits_Q (z+1)\,dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} (z+1)\,dzdydx$$