MATH 267 CHAPTER 17 MULTIPLE INTEGRALS

17.4 SURFACE AREA

Suppose $f(x,y) \geq 0$ throughout a region *R* in the *xy*-plane and *f* has continuous first partial derivatives on *R*.

Let *S* denote the portion of the graph of *f* whose projection on the *xy*-plane is *R*.

Assume no normal vector to *S* is parallel to the *xy*-plane.

Goal: Define the area *A* of *S* and find a formula for *A*.

Let $P = \{R_k\}$ be an inner partition of *R*, and let Δx_k and Δy_k be the dimensions of the rectangle R_k. For each *k*, choose any point $(x_k, y_k, 0)$ in R_k and let $B_k(x_k, y_k, f(x_k, y_k))$ be the corresponding point on *S*.

Next, consider the tangent plane to *S* at B_k and let ΔT_k and ΔS_k be the areas of the regions on the tangent plane and on *S*, resp., obtained by projecting R_k vertically upward. If the norm $\|P\|$ of the partition is small then ΔT_k is an approximation of ΔS_k, and $\sum_k \Delta T_k$ is an approximation to the area *A* of *S*. We then have

$$A = \lim_{\|P\| \to 0} \sum_k \Delta T_k .$$

Next, we wish to find an integral formula for *A*.

Choose $(x_k, y_k, 0)$ to be the corner of R_k closest to the origin. Consider the vectors $\vec{a}$ and $\vec{b}$ with initial point $B_k(x_k, y_k, f(x_k, y_k))$ which are tangent to the traces of *S* on the planes $y = y_k$ and $x = x_k$, resp. From Chapter 16 we know that the slopes of the lines determined by $\vec{a}$ and $\vec{b}$ in these planes are $f_x(x,y)$ and $f_y(x,y)$, resp. It follows that

$$\vec{a} = \Delta x_k \hat{i} + f_x(x_k, y_k) \Delta x_k k$$
$$\vec{b} = \Delta y_k \hat{j} + f_y(x_k, y_k) \Delta y_k k.$$

From Section 14.4, the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$ is $\|\vec{a} \times \vec{b}\|$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \Delta x_k & 0 & f_x(x_k, y_k)\Delta x_k \\ 0 & \Delta y_k & f_y(x_k, y_k)\Delta x_k \end{vmatrix}$$
$$= -f_x(x_k, y_k)\Delta x_k \Delta y_k \hat{i} - f_y(x_k, y_k)\Delta x_k \Delta y_k \hat{j} + \Delta x_k \Delta y_k k$$

Thus,

$$\Delta T_k = \|\vec{a} \times \vec{b}\| = \sqrt{\left[f_x(x_k, y_k)\right]^2 + \left[f_y(x_k, y_k)\right]^2 + 1}\, \Delta x_k \Delta y_k$$

and we have the following:

Integral for Surface Area $A = \iint_R \sqrt{\left[f_x(x,y)\right]^2 + \left[f_y(x,y)\right]^2 + 1}\, dA$

Note: The formula may also be used if $f(x,y) \le 0$ on R.

Exercises Set up an iterated double integral that can be used to find the surface area of the portion of the graph of the equation that lies over the region R in the xy-plane that has the given boundary. Use symmetry whenever possible.

2) $x^2 - y^2 + z^2 = 1$; the square with vertices $(0,1),(1,0),(-1,0),(0,-1)$

Solution:

$$x^2 - y^2 + z^2 = 1 \Rightarrow z = \sqrt{1 - x^2 + y^2}$$

$$z_x = -\frac{2x}{2\sqrt{1-x^2+y^2}} = -\frac{x}{\sqrt{1-x^2+y^2}}$$

$$z_y = \frac{y}{\sqrt{1-x^2+y^2}}$$

$$A = 4\int_0^1 \int_0^{1-x} \sqrt{\frac{x^2}{1-x^2+y^2} + \frac{y^2}{1-x^2+y^2} + 1}\, dydx = 4\int_0^1 \int_0^{1-x} \sqrt{\frac{x^2+y^2+1-x^2+y^2}{1-x^2+y^2}}\, dydx$$

Find the surface area…

6) $z = y^2$; the triangle with vertices $(0,0),\ (0,2),\ (2,2)$

Solution:

$$z = y^2 \Rightarrow z_x = 0 \text{ and } z_y = 2y$$

$$A = \int_0^2 \int_0^y \sqrt{0^2 + 4y^2 + 1}\, dxdy = \int_0^2 \int_0^y \sqrt{4y^2+1}\, dxdy = \int_0^2 x\sqrt{4y^2+1}\Big]_0^y\, dy$$

$$= \int_0^2 y\sqrt{4y^2+1}\, dy = \frac{1}{8}\left(4y^2+1\right)^{3/2} \cdot \frac{2}{3}\Big]_0^2 = \frac{1}{12}\left(17^{3/2} - 1\right)$$

8) Find the surface area of the first octant portion of the cylinder $y^2 + z^2 = 9$ that lies inside the cylinder $x^2 + y^2 = 9$.

Solution:

$$y^2 + z^2 = 9 \Rightarrow z = \sqrt{9 - y^2} \Rightarrow z_x = 0 \text{ and } z_y = -\frac{y}{\sqrt{9-y^2}}$$

$$A = \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{\frac{y^2}{9-y^2} + 1}\; dydx = \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{\frac{9}{9-y^2}}\; dydx$$

$$= \int_0^3 \left[3\sin^{-1}\frac{y}{3}\right]_0^{\sqrt{9-x^2}} dx = \int_0^3 3\sin^{-1}\frac{\sqrt{9-x^2}}{3}\, dx \text{...too hard}$$

so...

$$A = \int_0^3 \int_0^{\sqrt{9-y^2}} \frac{3}{\sqrt{9-y^2}}\; dxdy = \int_0^3 \frac{3x}{\sqrt{9-y^2}}\Bigg]_0^{\sqrt{9-y^2}} dy = \int_0^3 3dy = 3y\Big]_0^3 = 9$$

10) S is the part of the plane $z = y + 1$ that is inside the cylinder $x^2 + y^2 = 1$

Solution:

$$z = y + 1 \Rightarrow z_x = 0 \text{ and } z_y = 1$$

$$A = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1^2 + 1}\, dydx = \int_{-1}^{1} \sqrt{2}\left(\sqrt{1-x^2} + \sqrt{1-x^2}\right)dx \text{...hard!}$$

so...use polar coordinates...

$$A = \int_0^{2\pi}\int_0^1 \sqrt{2}\ rdrd\theta = \int_0^{2\pi} \sqrt{2}\ \left.\frac{r^2}{2}\right]_0^1 d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2}\ d\theta = \left.\frac{\sqrt{2}}{2}\theta\right]_0^{2\pi} = \sqrt{2}\pi$$

12) S is the first octant part of the hyperbolic paraboloid $z = x^2 - y^2$ that is inside the cylinder $x^2 + y^2 = 1$.

Solution:

$$z = x^2 - y^2 \Rightarrow z_x = 2x \text{ and } z_y = 2y$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4r^2 + 1}$$

$$A = \int_0^{\pi/4}\int_0^1 \sqrt{4r^2 + 1}\ rdrd\theta = \int_0^{\pi/4} \left.\frac{1}{8}\left(4r^2 + 1\right)^{3/2} \cdot \frac{2}{3}\right]_0^1 d\theta$$

$$= \int_0^{\pi/4} \left[\frac{1}{12}(5)^{3/2} - \frac{1}{12}\right] d\theta = \frac{1}{12}\left(5^{3/2} - 1\right)\frac{\pi}{4}$$