MATH 267 CHAPTER 17 MULTIPLE INTEGRALS

17.3 DOUBLE INTEGRALS IN POLAR COORDINATES

Consider the **elementary polar region** bounded by arcs of circles of radii r_1 and r_2 with centers at the origin, and by two rays from the origin.

$$\begin{aligned}
\text{area} = \Delta A &= \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta \\
&= \frac{1}{2}\left(r_2^2 - r_1^2\right)\Delta\theta \\
&= \underbrace{\frac{1}{2}\left(r_2 + r_1\right)}_{\bar{r}}\underbrace{\left(r_2 - r_1\right)}_{\Delta r}\Delta\theta
\end{aligned}$$

$$\Delta A = \bar{r}\Delta r\,\Delta\theta \Rightarrow dA = r dr d\theta$$

Next, consider the region R bounded by two rays that make positive angles α and β with the polar axis and by the graphs of two polar equations $r = g_1(\theta)$ and $r = g_2(\theta)$ where g_1 and g_2 are continuous functions and $g_1(\theta) \le g_2(\theta)$ for $\alpha \le \theta \le \beta$.

$$A = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} r dr d\theta$$

Next, if f is a continuous function of r and θ, then we have the following:

Evaluation Theorem $\lim_{\|P\|\to 0}\sum_k f(r_k,\theta_k)r_k\,\Delta r_k\,\Delta\theta_k = \iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)r\,dr\,d\theta$

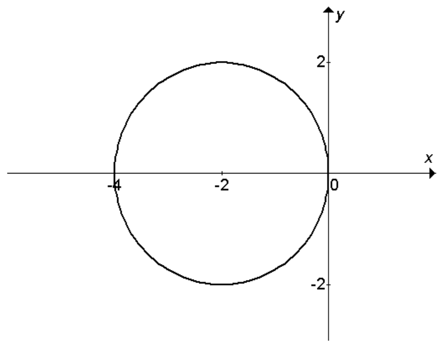
Change of variables formula $\iint_R f(x,y)\,dy\,dx = \iint_R f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$

Evaluation Theorem $\iint_R f(r,\theta)dA = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r,\theta)r\,d\theta\,dr$

Exercises

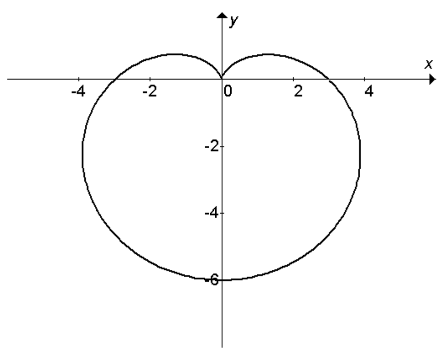
Express the area of the region as an iterated double integral in polar coordinates, using symmetry whenever possible.

2) region inside $r = -4\cos\theta$
Solution:



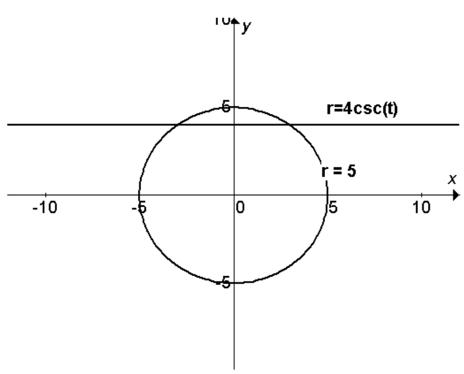
$$A = 2\int_0^{\pi/2}\int_0^{-4\cos\theta} r\,dr\,d\theta$$

4) region inside $r = 3 - 3\sin\theta$
Solution:



$$A = 2\int_{-\pi/2}^{\pi/2}\int_0^{3-3\sin\theta} r\,dr\,d\theta$$

6) see graph on p 910
Solution:

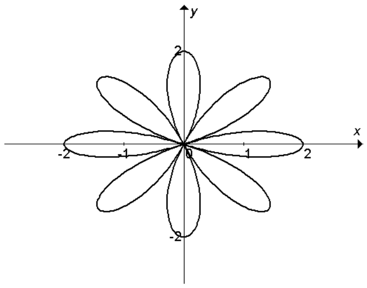


$$A = 2\int_{\arctan\frac{4}{3}}^{\pi/2}\int_{4\csc\theta}^{5} r\,dr\,d\theta$$

Use a double integral to find the area of the region that has the indicated shape.

8) one loop of $r = 2\cos 4\theta$

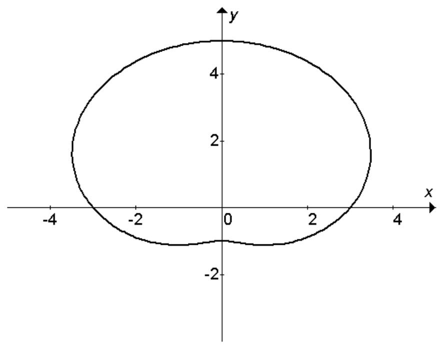
Solution:



$$A = 2\int_0^{\pi/8}\int_0^{2\cos 4\theta} r\,dr\,d\theta = 2\int_0^{\pi/8} \left.\frac{r^2}{2}\right]_0^{2\cos 4\theta} d\theta$$
$$= \int_0^{\pi/8} 4\cos^2 4\theta\, d\theta = 4\int_0^{\pi/8} \frac{1+\cos 8\theta}{2}\, d\theta$$
$$= 2\left[\theta + \frac{1}{8}\sin 8\theta\right]_0^{\pi/8} = 2\left[\frac{\pi}{8} + \frac{1}{8}\sin\pi - 0 - 0\right] = \frac{\pi}{4}$$

10) bounded by $r = 3 + 2\sin\theta$

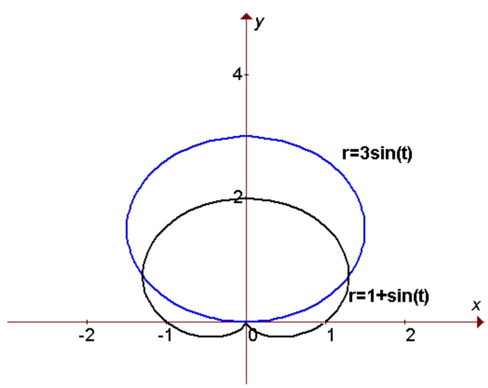
Solution:



$$A = \int_0^{2\pi}\int_0^{3+2\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(3+2\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(9+12\sin\theta+4\sin^2\theta\right)d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\left(9+12\sin\theta+4\cdot\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{1}{2}\left[9\theta-12\cos\theta+2\theta-\sin 2\theta\right]_0^{2\pi} = \frac{1}{2}\left[22\pi-12-0-(-12)\right] = 11\pi$$

12) inside $r = 3\sin\theta$ and outside $r = 1+\sin\theta$

Solution:

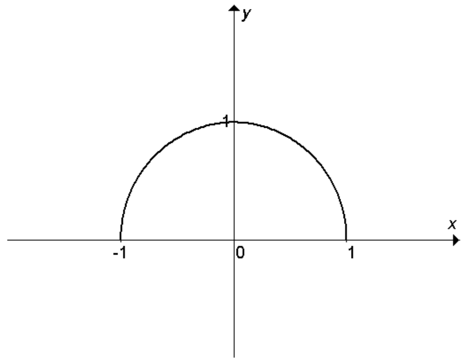


$$A = 2\int_{\pi/6}^{\pi/2}\int_{1+\sin\theta}^{3\sin\theta} r\,dr\,d\theta = 2\int_{\pi/6}^{\pi/2}\left.\frac{r^2}{2}\right]_{1+\sin\theta}^{3\sin\theta} d\theta \;= \int_{\pi/6}^{\pi/2}\left[9\sin^2\theta-(1+\sin\theta)^2\right]d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left[9\sin^2\theta-1-2\sin\theta-\sin^2\theta\right]d\theta \;= \int_{\pi/6}^{\pi/2}\left[8\sin^2\theta-1-2\sin\theta\right]d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left[8\cdot\frac{1-\cos 2\theta}{2}-1-2\sin\theta\right]d\theta = \left[4\theta-2\sin 2\theta-\theta+2\cos\theta\right]_{\pi/6}^{\pi/2} = \pi$$

Use polar coordinates to evaluate the integral.

14) $\iint_R x^2\left(x^2+y^2\right)^3 dA$; R is bounded by the semicircle $y=\sqrt{1-x^2}$ and the x – axis

Solution:



$$y=\sqrt{1-x^2} \Rightarrow y^2=1-x^2 \Rightarrow x^2+y^2=1 \Rightarrow r=1$$

$$x^2\left(x^2+y^2\right)^3=r^2\cos^2\theta\left(r^2\right)^3=r^8\cos^2\theta$$

$$\iint_R x^2\left(x^2+y^2\right)^3 dA=\int_0^{\pi}\int_0^1 r^8\cos^2\theta\cdot r\,dr\,d\theta=\int_0^{\pi}\left.\frac{r^{10}}{10}\cos^2\theta\right]_0^1 d\theta$$

$$=\int_0^{\pi}\frac{1}{10}\cos^2\theta\,d\theta=\frac{1}{10}\int_0^{\pi}\frac{1+\cos 2\theta}{2}\,d\theta=\frac{1}{10}\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi}=\frac{\pi}{20}$$

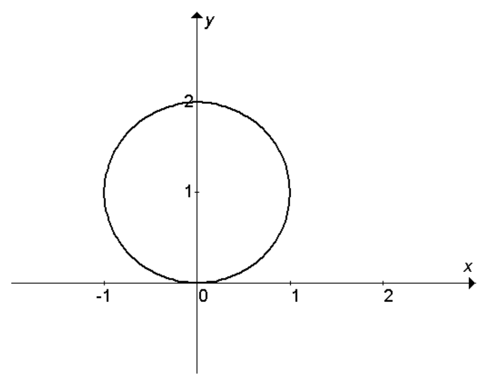
16) $\iint_R (x+y)\,dA$; R bounded by the circle $x^2+y^2=2y$

Solution:

$x^2+y^2=2y$

$x^2+y^2-2y+1=0+1$ OR $r^2=2r\sin\theta$

$x^2+(y-1)^2=1$ $\quad r=0$ or $r=2\sin\theta$



$$\iint_R (x+y)\,dA=\int_0^{\pi}\int_0^{2\sin\theta}(r\cos\theta+r\sin\theta)\,r\,dr\,d\theta$$

$$=\int_0^{\pi}\left[\frac{r^3}{3}\cos\theta+\frac{r^3}{3}\sin\theta\right]_0^{2\sin\theta}d\theta=\int_0^{\pi}\left[\frac{8}{3}\sin^3\theta\cos\theta+\frac{8}{3}\sin^4\theta\right]d\theta$$

$$=\int_0^{\pi}\left[\frac{8}{3}\sin^3\theta\cos\theta+\frac{8}{3}\left(\frac{1-\cos 2\theta}{2}\right)^2\right]d\theta=\int_0^{\pi}\left[\frac{8}{3}\sin^3\theta\cos\theta+\frac{8}{3}\cdot\frac{1}{4}\left(1-2\cos 2\theta+\cos^2 2\theta\right)\right]d\theta$$

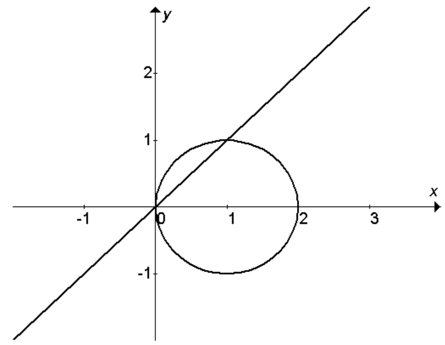
$$=\int_0^{\pi}\left[\frac{8}{3}\sin^3\theta\cos\theta+\frac{2}{3}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)\right]d\theta=\left[\frac{8}{3}\frac{\sin^4\theta}{4}+\frac{2}{3}\left(\frac{1}{2}\theta-\sin 2\theta+\frac{1}{8}\sin 4\theta\right)\right]_0^{\pi}=\frac{2\pi}{3}+\frac{\pi}{3}=\pi$$

18) $\iint_R \sqrt{x^2+y^2}\,dA$; R is bounded by the semicircle $y=\sqrt{2x-x^2}$ and the line $y=x$

Solution:

$$y=\sqrt{2x-x^2}$$
$$y^2=2x-x^2$$
$$x^2-2x+\mathbf{1}+y^2=\mathbf{1}$$
$$(x-1)^2+y^2=1$$
$$r=2\cos\theta$$

$$y=x$$
$$\theta=\frac{\pi}{4}$$



$$\iint_R \sqrt{x^2+y^2}\,dA=\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r\cdot r\,dr\,d\theta=\int_{\pi/4}^{\pi/2}\frac{r^3}{3}\Big]_0^{2\cos\theta}d\theta$$

$$=\int_{\pi/4}^{\pi/2}\frac{8}{3}\cos^3\theta\,d\theta=\frac{8}{3}\int_{\pi/4}^{\pi/2}\left(1-\sin^2\theta\right)\cos\theta\,d\theta=\frac{8}{3}\left[\sin\theta-\frac{\sin^3\theta}{3}\right]_{\pi/4}^{\pi/2}=\frac{8}{3}\left[1-\frac{1}{3}-\frac{\sqrt{2}}{2}+\frac{1}{3\sqrt{8}}\right]$$

Use polar coordinates to find the volume of the solid that has the shape of Q.

26) Q is cut out of the ellipsoid $4x^2+4y^2+z^2=16$ by the cylinder $x^2+y^2=1$.

Solution:

$$4x^2+4y^2+z^2=16\Rightarrow z=\sqrt{16-4x^2-4y^2}=\sqrt{16-4r^2}$$

$$V=8\int_0^{\pi/2}\int_0^1\sqrt{16-4r^2}\;r\,dr\,d\theta=\frac{8}{-8}\int_0^{\pi/2}\left(16-4r^2\right)^{3/2}\cdot\frac{2}{3}\Big]_0^1 d\theta$$

$$=-\frac{2}{3}\int_0^{\pi/2}\left[(16-4)^{3/2}-16^{3/2}\right]d\theta=-\frac{2}{3}\int_0^{\pi/2}\left[12^{3/2}-64\right]d\theta=-\frac{2}{3}\left[12^{3/2}\theta-64\theta\right]_0^{\pi/2}=\ldots$$

28) Q is bounded by the paraboloid $z=4x^2+4y^2$, the cylinder $x^2+y^2=3y$, and the plane $z=0$.

Solution:

$$x^2+y^2=3y\Rightarrow x^2+y^2-3y+\frac{9}{4}=\frac{9}{4}\Rightarrow x^2+\left(y-\frac{3}{2}\right)^2=\frac{9}{4}$$

$$r=3\sin\theta$$

$$z=4x^2+4y^2$$
$$z=4r^2$$

$$V=\int_0^{\pi}\int_0^{3\sin\theta}4r^2\cdot r\,dr\,d\theta=\int_0^{\pi}r^4\Big]_0^{3\sin\theta}d\theta=\int_0^{\pi}81\sin^4\theta\,d\theta=81\int_0^{\pi}\left(\frac{1-\cos2\theta}{2}\right)^2\,d\theta$$

$$=\frac{81}{4}\int_0^{\pi}\left(1-2\cos2\theta+\cos^2 2\theta\right)\,d\theta=\frac{81}{4}\int_0^{\pi}\left(1-2\cos2\theta+\frac{1+\cos4\theta}{2}\right)\,d\theta=\frac{81}{4}\left[\theta-\sin2\theta+\frac{1}{2}\theta+\frac{1}{8}\sin4\theta\right]_0^{\pi}$$

$$=\frac{81}{4}\left[\pi-0+\frac{\pi}{2}+0\right]=\frac{243}{8}\pi$$