MATH 267 CHAPTER 17 MULTIPLE INTEGRALS

17.1 DOUBLE INTEGRALS

Definition Let *f* be a function of two variables that is defined on a region *R*, and let $P=\{R_k\}$ be an inner partition of *R*. A **Riemann sum** of *f* for *P* is any sum of the form

$$\sum_k f(u_k,v_k)\,\Delta A_k\,,$$

where (u_k,v_k) is a point in R_k and ΔA_k is the area of R_k. The summation extends over all the subregions $R_1,R_2,\ldots,R_n$ of *P.*

Definition Let *f* be a function of two variables that is defined on a region *R*, and let *L* be a real number. The statement

$$\lim_{\|P\|\to 0}\sum_k f(u_k,v_k)\,\Delta A_k = L$$

means that for every $\in>0$ there is a $\delta>0$ such that if $P=\{R_k\}$ is an inner partition of *R* with $\|P\|<\delta$, then

$$\left|\sum_k f(u_k,v_k)\,\Delta A_k - L\right| < \in$$

for every choice of (u_k,v_k) in R_k.

In other words, we can make every Riemann sum of *f* as close as we want to *L* by choosing an inner partition of sufficiently small norm $\|P\|$.

Definition Let *f* be a function of two variables that is defined on a region *R*. The **double integral of *f* over** *R*, denoted by $\iint_R f(x,y)dA$, is

$$\iint_R f(x,y)dA = \lim_{\|P\|\to 0}\sum_k f(u_k,v_k)\,\Delta A_k\,,$$

provided the limit exists. If the double integral exists, then *f* is said to be **integrable over** *R*.

Definition Let *f* be a continuous function of two variables such that $f(x,y)\geq 0$ for every (x,y) in a region *R*. The **volume** *V* of the solid that lies under the graph of $z=f(x,y)$ and over *R* is

$$V = \iint_R f(x,y)dA\,.$$

Theorem (i) $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$ for every real number *c*

(ii) $\iint_R [f(x,y)+g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$

(iii) If *R* is the union of two nonoverlapping regions R_1 and R_2, then

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA\,.$$

(iv) If $f(x,y)\geq 0$ throughout *R*, then $\iint_R f(x,y)dA \geq 0$.

How to evaluate a double integral:

Definition (i) $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$

(ii) $\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$

Exercise

14) $\int_0^3 \int_{-2}^{-1} \left(4xy^3 + y\right) dxdy$

Solution:

$$\int_0^3 \int_{-2}^{-1} \left(4xy^3 + y\right) dxdy$$
$$= \int_0^3 \left(2x^2y^3 + xy\right)\Big]_{-2}^{-1} dy$$
$$= \int_0^3 \left[2(-1)^2 y^3 + (-1)y - \left(2(-2)^2 y^3 + (-2)y\right) \right] dy$$
$$= \int_0^3 \left(2y^3 - y - 8y^3 + 2y\right) dy$$
$$= \int_0^3 \left(-6y^3 + y\right) dy$$
$$= \left[-\frac{6y^4}{4} + \frac{y^2}{2} \right]_0^3$$
$$= -\frac{3(3)^4}{2} + \frac{(3)^2}{2} + 0 - 0 = -117$$

Suppose we change the order of integration in Exercise 14.

$$\int_{-2}^{-1} \int_0^3 \left(4xy^3 + y\right) dydx = \int_{-2}^{-1} \left[xy^4 + \frac{y^2}{2} \right]_0^3 dx$$
$$= \int_{-2}^{-1} \left[x(3)^4 + \frac{3^2}{2} - 0 - 0 \right] dx$$
$$= \left[\frac{81x^2}{2} + \frac{9}{2}x \right]_{-2}^{-1}$$
$$= \frac{81(-1)^2}{2} + \frac{9}{2}(-1) - \frac{81(-2)^2}{2} - \frac{9}{2}(-2)$$
$$= \frac{81}{2} - \frac{9}{2} - 162 + 9 = 36 - 162 + 9 = -117$$

Note: The order of integration does not matter.

Two Types of Regions

R_x region: $\left\{(x,y) \middle| a \le x \le b,\ g_1(x) \le y \le g_2(x)\right\}$

R_y region: $\left\{(x,y) \middle| h_1(y) \le x \le h_2(y),\ c \le y \le d\right\}$

Definition of iterated integrals

(i) $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$

(ii) $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right] dy$

Evaluation Theorem for double integrals

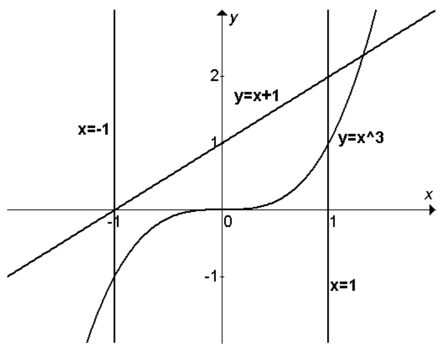
(i) $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

(ii) $\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Exercises

16) $\int_{-1}^{1} \int_{x^3}^{x+1} (3x+2y)\,dydx$

Solution:



$$\int_{-1}^{1} \int_{x^3}^{x+1} (3x+2y)\,dydx$$

$$= \int_{-1}^{1} \left[3xy + y^2 \right]_{x^3}^{x+1} dx$$

$$= \int_{-1}^{1} \left[3x(x+1) + (x+1)^2 - \left(3x \cdot x^3 + \left(x^3\right)^2 \right) \right]$$

$$= \int_{-1}^{1} \left(3x^2 + 3x + x^2 + 2x + 1 - 3x^4 - x^6 \right) dx$$

$$= \int_{-1}^{1} \left(4x^2 + 5x + 1 - 3x^4 - x^6 \right) dx$$

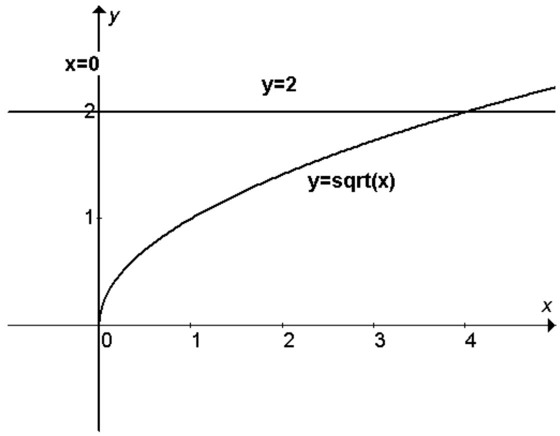
$$= \left[\frac{4x^3}{3} + \frac{5x^2}{2} + x - \frac{3x^5}{5} - \frac{x^7}{7} \right]_{-1}^{1}$$

$$= \frac{4}{3} + \frac{5}{2} + 1 - \frac{3}{5} - \frac{1}{7} - \left(-\frac{4}{3} + \frac{5}{2} - 1 + \frac{3}{5} + \frac{1}{7} \right)$$

Sketch the region *R* bounded by the graphs of the given equations. If $f(x,y)$ is an arbitrary continuous function, express $\iint_R f(x,y)\,dA$ as an iterated integral.

22) $y=\sqrt{x},\ \ x=0,\ \ y=2$

Solution:



As R_x region:

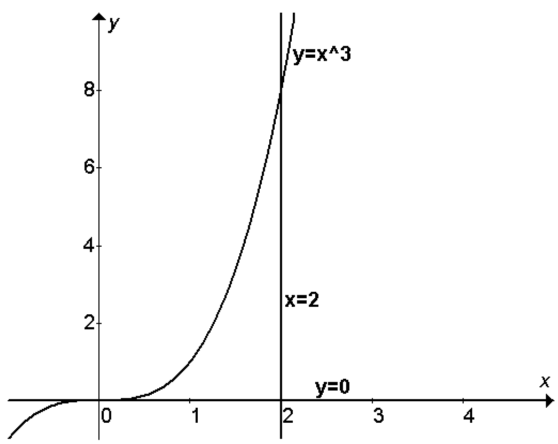
$$\iint_R f(x,y)\,dA=\int_0^4\int_{\sqrt{x}}^2 f(x,y)\,dydx$$

As R_y region:

$$\iint_R f(x,y)\,dA=\int_0^2\int_0^{y^2} f(x,y)\,dxdy$$

24) $y=x^3,\ \ x=2,\ \ y=0$

Solution:



As R_x region:

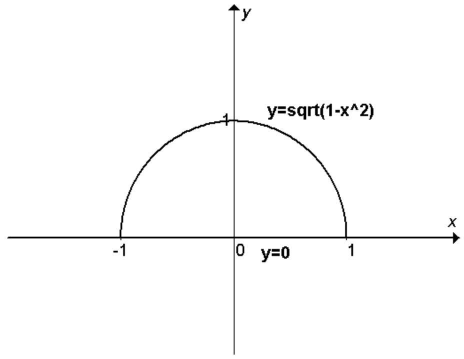
$$\iint_R f(x,y)\,dA=\int_0^2\int_0^{x^3} f(x,y)\,dydx$$

As R_y region:

$$\iint_R f(x,y)\,dA=\int_0^8\int_{\sqrt[3]{y}}^2 f(x,y)\,dxdy$$

26) $y=\sqrt{1-x^2},\;\; y=0$

Solution:



$R_x:\;\; 2\int_0^1\int_0^{\sqrt{1-x^2}} f(x,y)\,dydx$

or $\int_{-1}^1\int_0^{\sqrt{1-x^2}} f(x,y)\,dydx$

$R_y:\;\; 2\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dxdy$

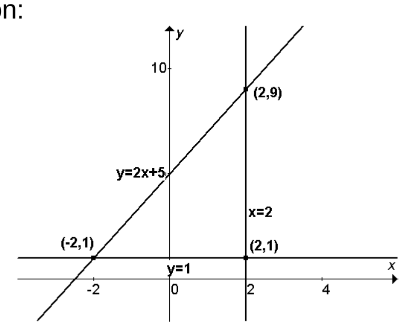
or $\int_0^1\int_{-\sqrt{1-y^2}}^0 f(x,y)\,dxdy+\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dxdy$

or $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dxdy$

Express the double integral over the indicated region R as an iterated integral and find its value.

28) $\iint\limits_R (x-y)\,dA$, over the triangular region with vertices $(2,9),\;(2,1),\;(-2,1)$

Solution:



$$\iint\limits_R (x-y)\,dA$$

$$=\int_{-2}^2\int_1^{2x+5}(x-y)\,dydx$$

$$=\int_{-2}^2\left[xy-\frac{y^2}{2}\right]_1^{2x+5}dx$$

$$=\int_{-2}^2\left[x(2x+5)-\frac{(2x+5)^2}{2}-x+\frac{1}{2}\right]dx$$

$$=\int_{-2}^2\left[2x^2+5x-\frac{1}{2}(4x^2+20x+25)-x+\frac{1}{2}\right]dx$$

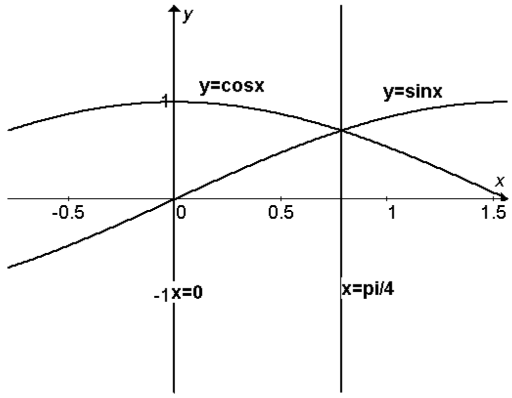
$$=\int_{-2}^2(-6x-12)\,dx=\left[-3x^2-12x\right]_{-2}^2$$

$$=\left[-3(2)^2-12(2)\right]-\left[-3(-2)^2-12(-2)\right]$$

$$=\left[-12-24\right]-\left[-12+24\right]=-36-12=-48$$

30) $\iint_R (y+1)\,dA$; R is the region between $y=\sin x$ and $y=\cos x$ from $x=0$ to $x=\pi/4$

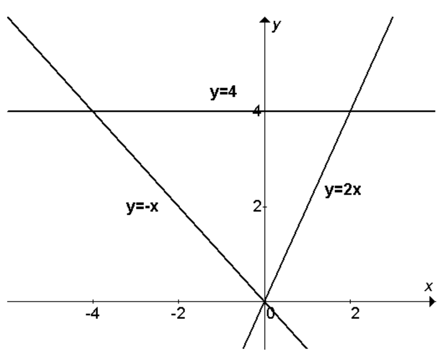
Solution:



$$\int_0^{\pi/4}\int_{\sin x}^{\cos x}(y+1)\,dydx$$
$$=\int_0^{\pi/4}\left[\frac{y^2}{2}+y\right]_{\sin x}^{\cos x}dx$$
$$=\int_0^{\pi/4}\left[\frac{\cos^2 x}{2}+\cos x-\frac{\sin^2 x}{2}-\sin x\right]dx$$
$$=\int_0^{\pi/4}\left[\frac{1}{2}\cdot\frac{1+\cos 2x}{2}+\cos x-\frac{1}{2}\cdot\frac{1-\cos 2x}{2}-\sin x\right]dx$$
$$=\left[\frac{1}{4}x+\frac{1}{8}\sin 2x+\sin x-\frac{1}{4}x+\frac{1}{8}\sin 2x+\cos x\right]_0^{\pi/4}$$
$$=\left[\frac{1}{4}\sin 2x+\sin x+\cos x\right]_0^{\pi/4}$$
$$=\frac{1}{4}\sin\frac{\pi}{2}+\sin\frac{\pi}{4}+\cos\frac{\pi}{4}-\left[\frac{1}{4}\sin 0+\sin 0+\cos 0\right]$$
$$=\frac{1}{4}+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-1=-\frac{3}{4}+\sqrt{2}$$

32) $\iint_R e^{x/y}\,dA$; $\quad R:\ y=2x,\ y=-x,\ y=4$

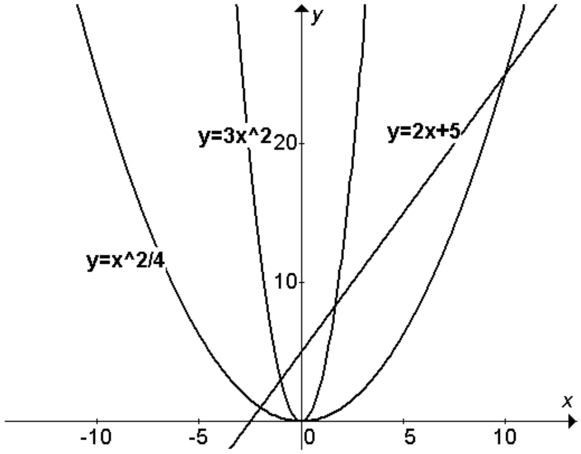
Solution:



$$R_y:\quad \int_0^4\int_{-y}^{y/2} e^{x/y}\,dxdy=\int_0^4 y\,e^{x/y}\Big]_{-y}^{y/2}dy=\int_0^4\left(ye^{1/2}-ye^{-1}\right)dy$$
$$=\left[e^{1/2}\frac{y^2}{2}-\frac{1}{e}\frac{y^2}{2}\right]_0^4=8\sqrt{e}-\frac{8}{e}$$

34) $x = 2\sqrt{y}, \quad \sqrt{3}x = \sqrt{y}, \quad y = 2x + 5$

Solution:

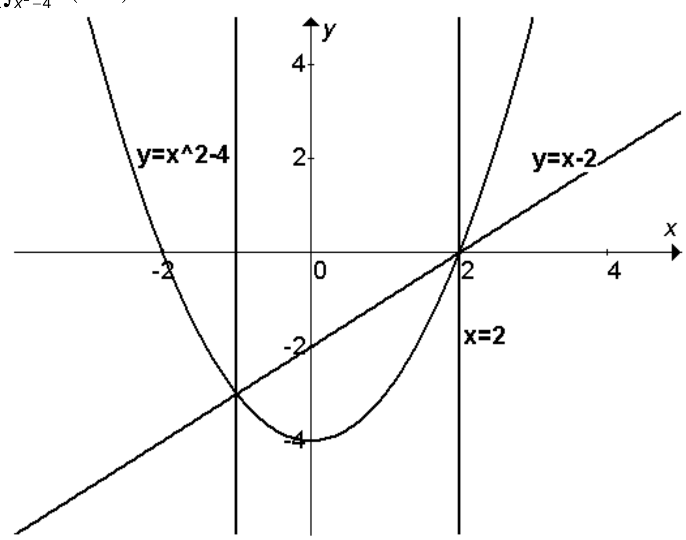


As R_x region: $\iint\limits_R f(x,y)\,dA = \int_0^{5/3}\int_{x^2/4}^{3x^2} f(x,y)\,dydx + \int_{5/3}^{10}\int_{x^2/4}^{2x+5} f(x,y)\,dydx$

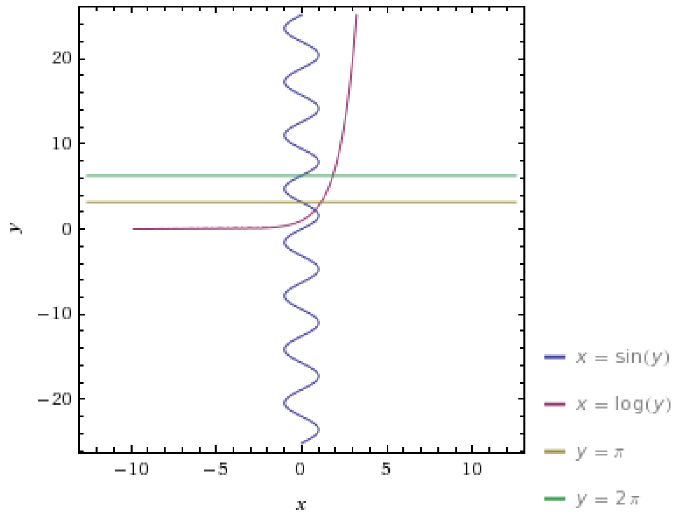
As R_y region: $\iint\limits_R f(x,y)\,dA = \int_0^{25/3}\int_{(y/3)^{1/2}}^{2\sqrt{y}} f(x,y)\,dydx + \int_{25/3}^{25}\int_{(y-5)/2}^{2\sqrt{y}} f(x,y)\,dydx$

Sketch the region of integration for the iterated integral.

40) $\int_{-1}^{2}\int_{x^2-4}^{x-2} f(x,y)\,dxdy$



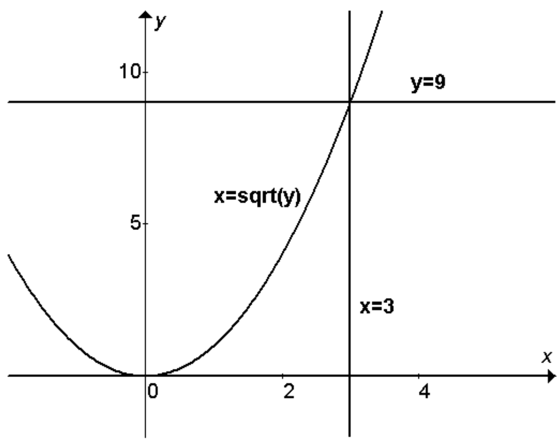
44) $\int_{\pi}^{2\pi}\int_{\sin y}^{\ln y} f(x,y)\,dxdy$



Reverse the order of integration and evaluate the resulting integral.

46) $\int_{0}^{9}\int_{\sqrt{y}}^{3} \sin x^3\,dxdy$

Solution:



$$\int_0^3\int_0^{x^2} \sin x^3\,dydx$$
$$=\int_0^3\left[\left(\sin x^3\right)y\right]_0^{x^2} dx$$
$$=\int_0^3 x^2 \sin x^3\,dx$$
$$=\frac{1}{3}\int_0^3 3x^2 \sin x^3\,dx$$
$$=-\frac{1}{3}\cos x^3\Big]_0^3$$
$$=-\frac{1}{3}\cos 27+\frac{1}{3}\cos 0=-\frac{1}{3}\cos 27+\frac{1}{3}$$