MATH 267 CHAPTER 16 PARTIAL DIFFERENTIATION

16.9 LAGRANGE MULTIPLIERS

Lagrange's Theorem Suppose f and g are functions of two variables that have continuous first partial derivatives, and that $\nabla g \neq \vec{0}$ throughout a region of the xy – plane. If f has an extremum $f(x_0, y_0)$ subject to the constraint $g(x,y)=0$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

The number λ is called a **Lagrange multiplier**.

Proof:

$g(x,y)=0$ is a curve C in the xy-plane.

$C: \quad x = h(t), \quad y = k(t), \quad t \in I$

Let $\vec{r}(t) = x\hat{i} + y\hat{j} = h(t)\hat{i} + k(t)\hat{j}$

$\vec{r}(t_0) = x_0\hat{i} + y_0\hat{j} = h(t_0)\hat{i} + k(t_0)\hat{j}$

Define $F(t) = f\left(h(t), k(t)\right)$. Then as t varies we get function values $f(x,y)$ that correspond to (x,y) on C, i.e. $f(x,y)$ is subject to the constraint $g(x,y)=0$. Since $f(x_0,y_0)$ is an extremum of f under these conditions, it follows that $F\left(h(t_0), k(t_0)\right)$ is an extremum of $F(t)$. Thus, $F'(t_0)=0$.

$$F'(t) = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} = f_x(x,y)h'(t) + f_y(x,y)k'(t)$$

Let $t = t_0$:

$$\begin{aligned} 0 = F'(t_0) &= f_x(x_0,y_0)h'(t_0) + f_y(x_0,y_0)k'(t_0) \\ &= \nabla f(x_0,y_0)\cdot \vec{r}\,'(t_0) \end{aligned}$$

$\Rightarrow \nabla f(x_0,y_0) \perp \vec{r}\,'(t_0) \quad \left(\perp \nabla g(x_0,y_0) \text{ since } C \text{ is level curve for } g\right)$

$\Rightarrow \nabla f \;//\; \nabla g$

$\Rightarrow \nabla f = \lambda \nabla g \quad$ for some λ

Corollary The points at which a function f of two variables has relative extrema subject to the constraint $g(x,y)=0$ are included among the points (x,y) determined by the first two coordinates of the solutions (x,y,λ) of the system of equations

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = 0 \end{cases}$$

Exercises

2) $f(x,y)=2x^2+xy-y^2+y;\;\; 2x+3y=1$

Solution:

$$f(x,y)=2x^2+xy-y^2+y$$
$$g(x,y)=2x+3y-1$$
$$\nabla f(x,y)=\lambda\nabla g(x,y)$$
$$\langle 4x+y,x-2y+1\rangle=\lambda\langle 2,3\rangle$$

Solve the system

$$\begin{cases} 4x+y=2\lambda \\ x-2y+1=3\lambda \\ 2x+3y=1 \end{cases} \Rightarrow \begin{cases} 4x+y-2\lambda=0 \\ x-2y-3\lambda=-1 \\ 2x+3y=1 \end{cases}$$

$2\cdot\text{first}+\text{second}$

$$8x+2y-4\lambda=0$$
$$\underline{x-2y-3\lambda=-1}$$
$$9x-7\lambda=-1\Rightarrow x=\frac{7\lambda-1}{9}$$

$-4\cdot\text{second}+\text{first}$

$$-4x+8y+12\lambda=4$$
$$\underline{4x+\;\;y-\;\;2\lambda=0}$$
$$9y+10\lambda=4\Rightarrow y=\frac{-10\lambda+4}{9}$$

Subsititute x and y in the 3rd equation:

$$2\cdot\frac{7\lambda-1}{9}+3\cdot\frac{-10\lambda+4}{9}=1$$
$$14\lambda-2-30\lambda+12=9$$
$$-16\lambda=-1\Rightarrow\lambda=1/16\Rightarrow x=\frac{7(1/16)-1}{9}=-\frac{1}{16}$$
$$y=\frac{-10(1/16)+4}{9}=\frac{3}{8}$$
$$f\left(-\frac{1}{16},\frac{3}{8}\right)=2\left(-\frac{1}{16}\right)^2+\left(-\frac{1}{16}\right)\left(\frac{3}{8}\right)-\left(\frac{3}{8}\right)^2+\frac{3}{8}=\frac{56}{256}\text{ is LMIN}$$
$$\text{since } f\left(0,\frac{1}{3}\right)=2(0)^2+(0)\left(\frac{1}{3}\right)-\left(\frac{1}{3}\right)^2+\frac{1}{3}=\frac{2}{3}>\frac{56}{256}$$

Extension of Lagrange's Theorem

Suppose that f and g are functions of three variables having continuous first partial derivatives and that $\nabla g\neq\vec{0}$ throughout a region an an xyz – coordinate system. If f has an extremum $f(x_0,y_0,z_0)$ subject to the constraint $g(x,y,z)=0$, then there is a real number λ such that

$$\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0).$$

Corollary The points at which a function *f* of three variables has relative extrema subject to the constraint $g(x,y,z)=0$ are included among the points (x,y,z) determined by the first three coordinates of the solutions (x,y,z,λ) of the system of equations

$$\begin{cases} f_x(x,y,z) = \lambda g_x(x,y,z) \\ f_y(x,y,z) = \lambda g_y(x,y,z) \\ f_z(x,y,z) = \lambda g_z(x,y,z) \\ g(x,y,z)=0 \end{cases}$$

Exercises

4) $f(x,y,z) = x^2 + y^2 + z^2;\ \ x + y + z = 25$

Solution:

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$g(x,y,z) = x + y + z - 25$$

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

Solve the system

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \\ x + y + z = 25 \end{cases}$$

Substitute $x = y = z = \dfrac{\lambda}{2}$ in the 4th equation:

$$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} = 25 \Rightarrow 3\lambda = 50 \Rightarrow \lambda = \frac{50}{3} \Rightarrow x = y = z = \frac{25}{3}$$

$$f\left(\frac{25}{3},\frac{25}{3},\frac{25}{3}\right) = \left(\frac{25}{3}\right)^2 + \left(\frac{25}{3}\right)^2 + \left(\frac{25}{3}\right)^2 = \frac{625}{3} \text{ is LMIN}$$

6) $f(x,y,z) = x + 2y - 3z;\ \ z = 4x^2 + y^2$

Solution:

$$f(x,y,z) = x + 2y - 3z$$

$$g(x,y,z) = 4x^2 + y^2 - z$$

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

$$\begin{cases} 1 = 8\lambda x \Rightarrow x = 1/24 \\ 2 = 2\lambda y \Rightarrow y = 1/3 \\ -3 = -\lambda \Rightarrow \lambda = 3 \\ 4x^2 + y^2 = z \Rightarrow 4\left(\dfrac{1}{24}\right)^2 + \left(\dfrac{1}{3}\right)^2 = z = \dfrac{17}{144} \end{cases}$$

$$f\left(\frac{1}{24},\frac{1}{3},\frac{17}{144}\right) = \frac{1}{24} + 2\left(\frac{1}{3}\right) - 3\left(\frac{17}{144}\right) = \frac{17}{48} \text{ is LMAX}$$

Some applications may involve more than one constraint.

Max (Min) $f(x,y,z)$ subject to

$$g(x,y,z)=0$$
$$h(x,y,z)=0$$

If *f* has an extremum subject to these constraints, then the following must be satisfied for some real numbers λ and μ:

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z).$$

8) $f(x,y,z)=z-x^2-y^2$

s.t. $x+y+z=1$

$x^2+y^2=4$

Solution:

$$\begin{cases} -2x=\lambda+2x\mu & (1) \\ -2y=\lambda+2y\mu & (2) \\ 1=\lambda & (3) \\ x+y+z=1 & (4) \\ x^2+y^2=4 & (5) \end{cases}$$

Subsititute (3) into (1):

$$-2x=1+2x\mu \Rightarrow x=-\frac{1}{2+2\mu}$$

Subsititute (3) into (2):

$$-2y=1+2y\mu \Rightarrow y=-\frac{1}{2+2\mu}$$

Thus, $x=y$. Substitute this in (5):

$$2x^2=4 \Rightarrow x=\pm\sqrt{2}=y \Rightarrow z=1\pm 2\sqrt{2}$$

$f\left(\sqrt{2},\sqrt{2},1-2\sqrt{2}\right)=-3-2\sqrt{2}$ LMIN

$f\left(-\sqrt{2},-\sqrt{2},1+2\sqrt{2}\right)=-3+2\sqrt{2}$ LMAX

Applications

12) Find the point on the line of intersection of the planes $x+3y-2z=11$ and $2x-y+z=3$ that is closest to the origin.

Solution:

Min $x^2+y^2+z^2$

subject to

$$x+3y-2z=11 \Rightarrow g(x,y,z)=x+3y-2z-11$$
$$2x-y+z=3 \Rightarrow h(x,y,z)=2x-y+z-3$$

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z)$$

Solve the system

$$2x=\lambda+2\mu \Rightarrow x=\frac{1}{2}(\lambda+2\mu)$$
$$2y=3\lambda-\mu \Rightarrow y=\frac{1}{2}(3\lambda-\mu)$$
$$2z=-2\lambda+\mu \Rightarrow z=\frac{1}{2}(-2\lambda+\mu)$$
$$x+3y-2z=11 \Rightarrow \frac{1}{2}(\lambda+2\mu)+\frac{3}{2}(3\lambda-\mu)-\frac{2}{2}(-2\lambda+\mu)=11$$
$$2x-y+z=3 \Rightarrow \frac{2}{2}(\lambda+2\mu)-\frac{1}{2}(3\lambda-\mu)+\frac{1}{2}(-2\lambda+\mu)=3$$

The last two equations simplify to

$$\begin{array}{ll} 14\lambda-3\mu=22 \longrightarrow & 14\lambda-3\mu=22 \\ -\lambda+2\mu=2 \xrightarrow{\times 14} & \underline{-14\lambda+28\mu=28} \\ & 25\mu=50 \Rightarrow \mu=2 \\ -\lambda+2(2)=2 \Rightarrow \lambda=2 & \end{array}$$

Substitute in the first 3 equations to get $x=3,\ y=2,\ z=-1$.

14) Prove that a closed rectangular box of fixed volume and minimal surface area is a cube.
Proof:

Min $A=2xy+2yz+2xz$

s.t.

$$V=xyz$$

$\nabla A=\lambda\nabla V$ gives

$$2y+2z=\lambda yz \quad (1)$$
$$2x+2z=\lambda xz \quad (2)$$
$$2y+2x=\lambda xy \quad (3)$$
$$xyz=V \quad (4)$$

Multiply (1) by x and (2) by y and subtract:

$$\begin{array}{l} 2xy+2xz=\lambda xyz \\ \underline{-(2xy+2yz=\lambda xyz)} \\ 2xz-2yz=0 \Rightarrow 2z(x-y)=0 \Rightarrow x=y \end{array}$$

Multiply (2) by y and (3) by z and subtract:

$$\begin{array}{l} 2xy+2yz=\lambda xyz \\ \underline{-(2xz+2yz=\lambda xyz)} \\ 2xy-2xz=0 \Rightarrow 2x(y-z)=0 \Rightarrow y=z \end{array}$$

16) Find the dimensions of the rectangular box of maximum volume that has three of its faces in the coordinate planes, one vertex at the origin, and another vertex in the first octant on the plane $2x+3y+5z=90$.

Solution:

Max $V = xyz$

s.t. $2x+3y+5z=90 \Rightarrow g(x,y,z)=2x+3y+5z-90$

$\nabla V = \lambda \nabla g$ gives

$$yz = 2\lambda \Rightarrow \lambda = \frac{yz}{2} \quad (1)$$

$$xz = 3\lambda \Rightarrow \lambda = \frac{xz}{3} \quad (2)$$

$$xy = 5\lambda \Rightarrow \lambda = \frac{xy}{5} \quad (3)$$

$$2x+3y+5z=90 \quad (4)$$

(1) and $(2) \Rightarrow \frac{yz}{2} = \frac{xz}{3} \Rightarrow y = \frac{2}{3}x$

(2) and $(3) \Rightarrow \frac{xz}{3} = \frac{xy}{5} \Rightarrow z = \frac{3}{5}y = \frac{2}{5}x$

Substitute in (4):

$$2x + 3\cdot\frac{2}{3}x + 5\cdot\frac{2}{5}x = 90 \Rightarrow 6x = 90 \Rightarrow x = 15$$

$$y = \frac{2}{3}(15) = 10$$

$$z = \frac{2}{5}(15) = 6$$