MATH 267 CHAPTER 16 PARTIAL DIFFERENTIATION

16.8 EXTREMA OF FUNCTIONS OF SEVERAL VARIABLES

A function f of two variables has a **local maximum** at (a,b) if there is an open disk R containing (a,b) such that $f(x,y) \le f(a,b)$ for every (x,y) in R. The local maxima correspond to the high points on the graph S of f. A **local minimum** at (c,d) is defined similarly.

A region in the xy – plane is **bounded** if it is a subregion of a closed disk. If f is continuous on a closed and bounded region R, then f has a **maximum** $f(a,b)$ and **minimum** $f(c,d)$ for some (a,b) and (c,d) in R; i.e.

$$f(c,d) \le f(x,y) \le f(a,b)$$

for every (x,y) in R.

Definition Let f be a function of two variables. A pair (a,b) is a **critical point** of f if either

(i) $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or

(ii) $f_x(a,b)$ or $f_y(a,b)$ does not exist.

Remark A maximum or minimum of a function may occur at a boundary point of its domain R. The investigation of such **boundary extrema** requires a separate procedure.

Example 1 p 862 Let $f(x,y) = 1 + x^2 + y^2$, with $x^2 + y^2 \le 4$. Find the extrema of f.
Solution: Solve the system of equations:

$$\begin{cases} 2x = 0 \\ 2y = 0 \end{cases}$$

to get $(0,0)$ for the critical point. Thus, $f(0,0) = 1$ is the only possible local extremum. Also, because

$$f(x,y) = 1 + x^2 + y^2 > 1 \quad \text{if} \quad (x,y) \ne (0,0),$$

f has a local minimum 1 at $(0,0)$ and this value is also the minimum value of f on R.

To find the possible boundary extrema, investigate the points that lie on the boundary of R. We see that any such point leads to the maximum value, for example, $f(2,0) = 5$.

Note: Not every critical point may lead to an extremum.

Example 2 p 863 If $f(x,y) = y^2 - x^2$ and the domain of f is $\mathbb{R}^2$, find the extrema of f.
Solution: Solve the system of equations

$$\begin{cases} -2x = 0 \\ 2y = 0 \end{cases}$$

to get the critical point $(0,0)$. The only possible local extremum is $f(0,0) = 0$. Note however that if $y \ne 0$ then $f(0,y) = y^2 > 0$

if $x \ne 0$ then $f(x,0) = -x^2 < 0$.

Thus, every open disk in the *xy*-plane contains pairs at which the function values are greater than $f(0,0)$ and also pairs at which values are less than $f(0,0)$. Thus *f* has no local extrema.

To determine the extrema of a more complicated function *f* of two variables, use the following:

Definition Let *f* be a function of two variables that has continuous second partial derivatives. The **discriminant** *D* of *f* is given by

$$D(x,y)=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=f_{xx}(x,y)f_{yy}(x,y)-\left[f_{xy}(x,y)\right]^2.$$

Test for local extrema

Let *f* be a function of two variables that has continuous second partial derivatives throughout an open disk *R* containing (a,b). If $f_x(a,b)=f_y(a,b)=0$ and $D(a,b)>0$, then

- **(i)** $f_{xx}(a,b)<0 \Rightarrow f(a,b)$ is a local maximum
- **(ii)** $f_{xx}(a,b)>0 \Rightarrow f(a,b)$ is a local minimum

Note:

$$\begin{aligned} D(a,b)>0 &\Rightarrow f_{xx}(a,b)f_{yy}(a,b)>\left[f_{xy}(a,b)\right]^2 \\ &\Rightarrow f_{xx}(a,b)f_{yy}(a,b)>0 \\ &\Rightarrow f_{xx}(a,b) \text{ and } f_{yy}(a,b) \text{ have the same sign} \\ &\Rightarrow f_{xx}(a,b) \text{ may be replaced by } f_{yy}(a,b) \text{ in (i) and (ii)} \end{aligned}$$

Def A point $P(a,b,f(a,b))$ on the graph of *f* is a **saddle point** if $f_x(a,b)=f_y(a,b)=0$ and if there is an open disk *R* containing (a,b) such that $f(x,y)>f(a,b)$ for some points (x,y) in *R* and $f(x,y)<f(a,b)$ for other points.

Theorem Let *f* have continuous second partial derivatives throughout an open disk *R* containing (a,b). If $f_x(a,b)=f_y(a,b)=0$ and $D(a,b)<0$, then the point $P(a,b,f(a,b))$ is a saddle point on the graph of *f*.

Note: We cannot determine whether $f(a,b)$ is a local extremum if $D(a,b)=0$ or if either $f_x(a,b)$ or $f_y(a,b)$ does not exist. In these cases one must analyze the variation of $f(x,y)$ near (a,b) by referring to the definition of *f* or the graph of *f*. For example,

$$f(x,y)=-\left(x^2+y^{2/3}\right)$$

$$f_y(x,y)=-\frac{2}{3}y^{-1/3} \text{ DNE if } y=0 \Rightarrow \text{every point } (a,0) \text{ on the } x\text{-axis is c.p.}$$

Exercises Find the extrema and saddle points of *f.*

2) $f(x,y) = x^2 - 2x + y^2 - 6y + 12$

Solution:

$$\left.\begin{array}{l} f_x(x,y) = 2x - 2 = 0 \Rightarrow x = 1 \\ f_y(x,y) = 2y - 6 = 0 \Rightarrow y = 3 \end{array}\right\} \Rightarrow (1,3) \ \text{ c.p.}$$

$$\left.\begin{array}{l} f_{xx}(x,y) = 2 \\ f_{yy}(x,y) = 2 \\ f_{xy}(x,y) = 0 \end{array}\right\} \Rightarrow D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = 2(2) - 0^2 > 0$$

Since $f_{xx}(1,3) = 2 > 0$ and $D(1,3) = 4 > 0,$ then $f(1,3) = 1^2 - 2(1) + 3^2 - 6\cdot 3 + 12 = 2$ is a local min.

f has no saddle points.

4) $f(x,y) = 5 + 4x - 2x^2 + 3y - y^2$

Solution:

$$\left.\begin{array}{l} f_x(x,y) = 4 - 4x = 0 \Rightarrow x = 1 \\ f_y(x,y) = 3 - 2y = 0 \Rightarrow y = 3/2 \end{array}\right\} \Rightarrow (1,3/2) \ \text{ c.p.}$$

$$\left.\begin{array}{l} f_{xx}(x,y) = -4 \\ f_{yy}(x,y) = -2 \\ f_{xy}(x,y) = 0 \end{array}\right\} \Rightarrow D(x,y) = 8 > 0$$

$$f_{xx}(1,3/2) < 0 \Rightarrow f(1,3/2) = 5 + 4 - 2 + 3(3/2) - (3/2)^2 = 37/4 \text{ is a local max}$$

8) $f(x,y) = x^2 + xy$

Solution:

$$\left.\begin{array}{l} f_x(x,y) = 2x + y = 0 \\ f_y(x,y) = x = 0 \end{array}\right\} \Rightarrow (0,0) \ \text{c.p.}$$

$$\left.\begin{array}{l} f_{xx}(x,y) = 2 \\ f_{yy}(x,y) = 0 \\ f_{xy}(x,y) = 1 \end{array}\right\} \Rightarrow D(x,y) = 0 - 1^2 = -1 < 0$$

Thus, $(0,0,0)$ is a saddle point.

12) $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{3}{2}x^2 - 4y$

Solution:

$$f_x(x,y) = x^2 - 3x = 0 \Rightarrow x(x-3) = 0 \Rightarrow x = 0,3$$

$$f_y(x,y) = y^2 - 4 = 0 \Rightarrow y = \pm 2$$

Thus, $(0,2),\ (0,-2),\ (3,2),\ (3,-2)$ are c.p.

$$\left.\begin{array}{l} f_{xx}(x,y) = 2x - 3 \\ f_{yy}(x,y) = 2y \\ f_{xy}(x,y) = 0 \end{array}\right\} \Rightarrow D(x,y) = 4xy - 6y$$

at $(0,2)$: $D(0,2) = -12 < 0 \Rightarrow (0,2,f(0,2))$ saddle point

at $(0,-2)$: $D(0,-2) = 12 > 0$ and $f_{xx}(0,-2) = 0 - 3 < 0 \Rightarrow f(0,-2) = 16/3$ local max

at $(3,2)$: $D(3,2) = 24 - 12 > 0$ and $f_{yy}(3,2) = 4 > 0 \Rightarrow f(3,2) = -59/6$ local min

at $(3,-2)$: $D(3,-2) = -24 + 12 < 0 \Rightarrow (3,-2,f(3,-2))$ saddle point

18) $f(x,y) = x \sin y$

Solution:

$$\left.\begin{aligned} f_x(x,y) &= \sin y = 0 \Rightarrow y = n\pi,\ n \text{ integer} \\ f_y(x,y) &= x\cos y = 0 \Rightarrow x = 0 \end{aligned}\right\} \Rightarrow (0,n\pi) \text{ c.p.}$$

$$\left.\begin{aligned} f_{xx}(x,y) &= 0 \\ f_{yy}(x,y) &= -x\sin y \\ f_{xy}(x,y) &= \cos y \end{aligned}\right\} \Rightarrow D(x,y) = 0 - \cos^2 y = -1 < 0 \text{ for } y = n\pi,\ n \text{ integer}$$

Thus, $(0,n\pi,0)$ are saddle points.

Finding the max and min on a bounded region *R*:

24) $f(x,y) = 5 + 4x - 2x^2 + 3y - y^2$; R is the triangular region bounded by the lines $y = x$, $y = -x$, and $y = 2$

Solution: In Exercise 4, we found that f has a local max at $(1,3/2)$ which is within R. So we need to check only for boundary extrema.

Let $C_1 : y = x$, $C_2 : y = -x$, $C_3 : y = 2$

On $C_1 : y = x$

$$\begin{aligned} f(x,y) = f(x,x) &= 5 + 4x - 2x^2 + 3x - x^2 \\ h(x) &= 5 + 7x - 3x^2 \\ h'(x) &= 7 - 6x = 0 \Rightarrow x = 7/6 \\ h(7/6,7/6) &= 109/12 \end{aligned}$$

On $C_3 : y = 2$

$$\begin{aligned} f(x,y) = f(x,2) &= 5 + 4x - 2x^2 + 6 - 4 \\ h(x) &= 7 + 4x - 2x^2 \\ h'(x) &= 4 - 4x = 0 \Rightarrow x = 1 \\ f(1,2) &= -2 + 4 + 7 = 9 \end{aligned}$$

On $C_2 : y = -x$

$$\begin{aligned} f(x,y) = f(x,-x) &= 5 + 4x - 2x^2 - 3x - x^2 \\ h(x) &= 5 + x - 3x^2 \\ h'(x) &= 1 - 6x = 0 \Rightarrow x = 1/6 \\ f(1/6,-1/6) &= 5 + 1/6 - 3(1/6)^2 = 61/12 \end{aligned}$$

At the corners:

$$\begin{aligned} f(0,0) &= 5 \\ f(2,2) &= 5 + 4\cdot 2 - 2\cdot 2^2 + 3\cdot 2 - 2^2 = 7 \\ f(-2,2) &= 5 + 4(-2) - 2(-2)^2 + 3\cdot 2 - 2^2 = -9 \end{aligned}$$

Thus, $MIN = f(-2,2) = -9$ and $MAX = f(1,3/2) = 37/4$.

Applications

30) Find the shortest distance between the parallel planes $2x+3y-z=2$ and $2x+3y-z=4$.

Solution:

$\pi_1:\ 2x+3y-z=2$

$\pi_2:\ 2x+3y-z=4$ or $z=2x+3y-4$

Take a point $P(0,0,-2)$ on π_1. Let $f(x,y)=x^2+y^2+(2x+3y-2)^2$ give the square of the distance between P and π_2.

$$f(x,y)=x^2+y^2+(2x+3y-2)^2$$
$$f_x(x,y)=2x+2(2x+3y-2)\cdot 2=10x+12y-8=0 \quad (1)$$
$$f_y(x,y)=2y+2(2x+3y-2)\cdot 3=12x+20y-12=0 \quad (2)$$

$$(1)\quad 5x+6y=4 \xrightarrow{\times 5} 25x+30y=20$$
$$(2)\quad 3x+5y=3 \xrightarrow{\times(-6)} \underline{-18x-30y=-18}$$
$$7x \qquad =2 \Rightarrow x=2/7,\ y=3/7$$

$(2/7,3/7)$ c.p.

$f_{xx}=10,\ f_{yy}=20,\ f_{xy}=12,\ D=200-12^2>0 \Rightarrow d=\sqrt{14}/7$ loc min

32) Find three positive real numbers whose sum is 1000 and whose product is a maximum.

Solution: Find x,y,z such that $x+y+z=1000$ and $P=xyz$ is a maximum.

Let

$$f(x,y)=xy(1000-x-y)=1000xy-x^2y-xy^2$$
$$f_x(x,y)=1000y-2xy-y^2=0$$
$$\Rightarrow 1000-2x-y=0 \Rightarrow 2x+y=1000 \quad (1)$$
$$f_y(x,y)=1000x-x^2-2xy=0$$
$$\Rightarrow 1000-x-2y=0 \Rightarrow x+2y=1000 \quad (2)$$

Solving (1) and (2) we get

$x=1000/3,\quad y=1000/3,\quad z=1000/3$

$f_{xx}=-2y<0$

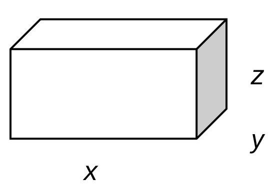
$f_{yy}=-2x<0$

$f_{xy}=1000-2x-2y$

$D>0$

34) If an open rectangular box is to have a fixed surface area A, what relative dimensions will make the volume a maximum?

Solution:



$$A=2xz+2yz+xy \Rightarrow A-xy=(2x+2y)z \Rightarrow z=\frac{A-xy}{2x+2y}$$

$$V(x,y)=\frac{xy(A-xy)}{2x+2y} \qquad \text{Let } A=1.$$

$$V(x,y)=\frac{xy(1-xy)}{2x+2y}=\frac{xy-x^2y^2}{2x+2y}$$

$$V_x(x,y)=\frac{(2x+2y)\left(y-2xy^2\right)-\left(xy-x^2y^2\right)2}{(2x+2y)^2}$$

$$=\frac{2xy-4x^2y^2+2y^2-2xy^3-2xy+2x^2y^2}{(2x+2y)^2}$$

$$=\frac{-2x^2y^2-4xy^3+2y^2}{(2x+2y)^2}=0\Rightarrow -x^2-2xy+1=0 \quad (1)$$

$$V_y(x,y)=\frac{(2x+2y)\left(x-2x^2y\right)-\left(xy-x^2y^2\right)2}{(2x+2y)^2}=0$$

$$\Rightarrow 2x^2+2xy-4x^3y-4x^2y^2-2xy+2x^2y^2=0$$

$$\Rightarrow 1-2xy-y^2=0 \quad (2)$$

Solving (1) and (2):

$$\left.\begin{array}{l} -x^2-2xy+1=0\Rightarrow 1-x^2=2xy \\ 1-2xy-y^2=0\Rightarrow 1-y^2=2xy \end{array}\right\}\Rightarrow x^2=y^2\Rightarrow x=y$$

$$1-2y^2-y^2=0\Rightarrow 1-3y^2=0\Rightarrow y=1/\sqrt{3}=x$$

$$z=\frac{1-1/3}{2/\sqrt{3}+2/\sqrt{3}}=\frac{2}{3}\cdot\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{6}$$

40) A window has the shape of a rectangle surmounted by an isosceles triangle. If the perimeter of the window is 12 feet, what values of x, y, and θ will maximize the total area?

Solution:

$$\text{Max } A=xy+\frac{1}{2}x\left(\frac{1}{2}x\tan\theta\right)=xy+\frac{1}{4}x^2\tan\theta$$

Now

$$P=x+2y+x\sec\theta=12\Rightarrow y=\frac{1}{2}(12-x-x\sec\theta)$$

Substitute in A:

$$A=x\frac{1}{2}(12-x-x\sec\theta)+\frac{1}{4}x^2\tan\theta$$

$$A=6x-\frac{1}{4}x^2(2+2\sec\theta-\tan\theta)$$

$$A_\theta=-\frac{1}{4}x^2\left(2\sec\theta\tan\theta-\sec^2\theta\right)=0\Rightarrow 2\tan\theta=\sec\theta$$

$$\Rightarrow 2\sin\theta=1\Rightarrow \theta=\pi/6$$

$$A_x=6-\frac{1}{2}x(2+2\sec\theta-\tan\theta)=6-\frac{1}{2}x\left(2+\frac{4}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)$$

$$=6-\frac{1}{2}x\left(2+\sqrt{3}\right)=0$$

$$\Rightarrow x=\frac{12}{2+\sqrt{3}}=24-12\sqrt{3}$$

$$y=6-2\sqrt{3}$$