MATH 267 CHAPTER 16 PARTIAL DIFFERENTIATION

16.7 TANGENT PLANES AND NORMAL LINES

Suppose a surface *S* is the graph of an equation $F(x,y,z)=0$ and *F* has continuous first partial derivatives. Let $P_0(x_0,y_0,z_0)$ be a point on *S* at which F_x, F_y, and F_z are not all zero. A **<u>tangent line</u>** to *S* at P_0 is a tangent line *l* to any curve *C* that lies on *S* and contains P_0. If *C* has the parametrization

$$x=f(t), \quad y=g(t), \quad z=h(t)$$

for *t* in some interval *I* and if $\vec{r}(t)$ is the position vector of $P(x,y,z)$, then

$$\vec{r}(t)=\langle f(t),g(t),h(t)\rangle .$$

Hence

$$\vec{r}\,'(t)=\langle f'(t),g'(t),h'(t)\rangle$$

is a tangent vector to *C* at $P(x,y,z)$.

For each *t*, the point $(f(t),g(t),h(t))$ on *C* is also on *S*, and therefore

$$F\big(f(t),\, g(t),\, h(t)\big)=0 .$$

If we let

$$w=F(x,y,z) \text{ , with } x=f(t),\ y=g(t),\ z=h(t) ,$$

then using a chain rule and the fact that $w=0$ for every *t*, we have

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}=0 .$$

Thus, for every point $P(x,y,z)$ on *C*,

$$F_x(x,y,z)f'(t)+F_y(x,y,z)g'(t)+F_z(x,y,z)h'(t)=0 ,$$

or, equivalently,

$$\nabla F(x,y,z)\cdot\vec{r}\,'(t)=0 .$$

In particular, if $P_0(x_0,y_0,z_0)$ corresponds to $t=t_0$, then

$$\nabla F\big]_{P_0}\cdot\vec{r}\,'(t_0)=\nabla F(x_0,y_0,z_0)\cdot\vec{r}\,'(t_0)=0 .$$

Since $\vec{r}\,'(t_0)$ is a tangent vector to *C* at P_0, this implies that the vector $\nabla F\big]_{P_0}$ is orthogonal to every tangent line *l* to *S* at P_0.

The plane through P_0 with normal vector $\nabla F\big]_{P_0}$ is the **<u>tangent plane</u> to *S* at P_0.**

Thus, every tangent line *l* to *S* at P_0 lies in the tangent plane at P_0.

Theorem Suppose $F(x,y,z)$ has continuous first partial derivatives and S is the graph of $F(x,y,z)=0$. If P_0 is a point on S and if $F_x, F_y,$ and F_z are not all 0 at P_0, then the vector $\nabla F]_{P_0}$ is normal to the tangent plane to S at P_0.

$\nabla F]_{P_0}$ is referred to as a vector **normal to the surface** S at P_0.

Corollary An equation for the tangent plane to the graph of $F(x,y,z)=0$ at the point $P_0(x_0,y_0,z_0)$ is

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.$$

If $z=f(x,y)$ is an equation for S and we let $F(x,y,z)=f(x,y)-z$, then the equation in the above corollary becomes

$$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+(-1)(z-z_0)=0$$

and we get:

Theorem An equation for the tangent plane to the graph of $z=f(x,y)$ at the point (x_0,y_0,z_0) is

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0).$$

The line perpendicular to the tangent plane at a point $P_0(x_0,y_0,z_0)$ on a surface S is a **normal line** to S at P_0. If S is the graph of $F(x,y,z)=0$, then the normal line is parallel to the vector $\nabla F(x_0,y_0,z_0)$.

Exercises Find equations for the tangent plane and the normal line to the graph of the equation at P.

2) $9x^2-4y^2-25z^2=40;\quad P(4,1,-2)$

Solution:

$F(x,y,z)=9x^2-4y^2-25z^2-40=0$

$F_x(x,y,z)=18x,\quad F_y(x,y,z)=-8y,\quad F_y(x,y,z)=-50z$

at $(4,1,-2)$:

$F_x(4,1,-2)=72,\quad F_y(4,1,-2)=-8,\quad F_y(4,1,-2)=100$

Applying the Corollary, we get:

tangent plane: $72(x-4)-8(y-1)+100(z+2)=0$

normal line: $x=4+72t,\quad y=1-8t,\quad z=-2+100t,\quad t\in\mathbb{R}$

4) $z=4x^2-y^2;\quad P(5,-8,36)$

Solution:

$f(x,y)=4x^2-y^2$

$f_x(x,y)=8x,\qquad f_y(x,y)=-2y$

$f_x(5,-8)=40,\qquad f_y(5,-8)=16$

tangent plane: $40(x-5)+16(y+8)-(z-36)=0$

normal line: $x=5+40t,\quad y=-8+16t,\quad z=36-t,\quad t\in\mathbb{R}$

Geometric Interpretation for the Differential *dz*

Suppose *S* is the graph of the equation $z = f(x,y)$ and $P_0(x_0,y_0,z_0)$ is on *S*. Since

$$\Delta z = f(x_0+\Delta x,\ y_0+\Delta y) - f(x_0,y_0)$$

the point $P(x_0+\Delta x,\ y_0+\Delta y,\ z_0+\Delta z)$ is on *S*.

Let $C(x_0+\Delta x,\ y_0+\Delta y,\ z)$ be on the tangent plane, and consider the points $A(x_0+\Delta x,\ y_0+\Delta y,\ 0)$ and $B(x_0+\Delta x,\ y_0+\Delta y,\ z_0)$. Since *C* is on the tangent plane, its coordinates satisfy the equation in Theorem 16.35, i.e.

$$z - z_0 = f_x(x_0,y_0)(x_0+\Delta x - x_0) + f_y(x_0,y_0)(y_0+\Delta y - y_0) = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y = dz$$

We have shown that dz is the distance from *B* to the point on the tangent plane that lies above or below *B*.

Discuss Figure 16.63 p 858

Theorem Let a function *F* of three variables be differentiable at $P_0(x_0,y_0,z_0)$, and let *S* be the level surface of *F* containing P_0. If $\nabla F(x_0,y_0,z_0) \neq \vec{0}$, then this gradient vector is normal to *S* at P_0. Thus, the direction for the maximum rate of change of $F(x,y,z)$ at P_0 is normal to *S.*

The following result for functions of two variables is analogous to the above:

Theorem Let a function *f* of two variables be differentiable at $P_0(x_0,y_0)$, and let *C* be the level curve of *f* that contains P_0. If $\nabla f(x_0,y_0) \neq \vec{0}$, then this gradient vector is orthogonal to *C* at P_0. Thus, the direction for the maximum rate of change of $f(x,y)$ at P_0 is orthogonal to *C*.

Exercises Sketch both the level curve *C* of *f* that contains *P* and $\nabla f\big]_P$.

12) $f(x,y) = 3x - 2y;\quad P(-2,1)$

Solution: $f(-2,1) = 3(-2) - 2(1) = -8$

The level curve that passes through $(-2,1)$ is $3x - 2y = -8$.

$\nabla f(-2,1) = \langle 3,-2\rangle$

14) $f(x,y) = xy;\quad P(3,2)$

Solution: $f(3,2) = 6 \Rightarrow xy = 6 \Rightarrow y = \dfrac{6}{x}$

$\nabla f(3,2) = \langle 2,3\rangle$

Sketch both the level surface *S* of *F* that contains *P* and $\nabla f\big]_P$.

16) $F(x,y,z) = z - x^2 - y^2;\quad P(2,-2,1)$

Solution:

$F(2,-2,1) = 1 - 2^2 - (-2)^2 = -7 \Rightarrow z - x^2 - y^2 = -7 \Rightarrow z + 7 = x^2 + y^2$

$\nabla F(x,y,z) = \langle -2x,-2y,1\rangle$

$\nabla F(2,-2,1) = \langle -4,4,1\rangle$

Prove that an equation of the tangent plane to the given quadric surface at the point $P_0(x_0, y_0, z_0)$ may be written in the indicated form.

22) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$; $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} + \dfrac{zz_0}{c^2} = 1$

Solution: $f(x,y,z) = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1 = 0$

$$f_x(x_0,y_0,z_0) = \frac{2x_0}{a^2}; \quad f_y(x_0,y_0,z_0) = -\frac{2y_0}{b^2}; \quad f_z(x_0,y_0,z_0) = \frac{2z_0}{c^2}$$

tangent plane: $\dfrac{2x_0}{a^2}(x - x_0) - \dfrac{2y_0}{b^2}(y - y_0) + \dfrac{2z_0}{c^2}(z - z_0) = 0$

$$\frac{xx_0}{a^2} - \frac{x_0^2}{a^2} - \frac{yy_0}{b^2} + \frac{y_0^2}{b^2} + \frac{zz_0}{c^2} - \frac{z_0^2}{c^2} = 0$$

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}$$

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} + \frac{zz_0}{c^2} == 1$$

26) Show that the sum of the squares of the *x*-, *y*-, and *z*-intercepts of every tangent plane to the graph of the equation $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ is the constant a^2.

Solution: $f_x(x,y,z) = \dfrac{2}{3x^{1/3}}$, $\quad f_y(x,y,z) = \dfrac{2}{3y^{1/3}}$, $\quad f_z(x,y,z) = \dfrac{2}{3z^{1/3}}$

tangent plane: $\dfrac{1}{x_0^{1/3}}(x - x_0) + \dfrac{1}{y_0^{1/3}}(y - y_0) + \dfrac{1}{z_0^{1/3}}(z - z_0) = 0$

Let A, B, and C be the $x-, y-$, and $z-$intercepts, resp.

$x-$int: set $y = 0$ and $z = 0$

$$\frac{1}{x_0^{1/3}}(A - x_0) + \frac{1}{y_0^{1/3}}(0 - y_0) + \frac{1}{z_0^{1/3}}(0 - z_0) = 0$$

$$\frac{1}{x_0^{1/3}}(A - x_0) - y_0^{2/3} - z_0^{2/3} = 0$$

$$A - x_0 = x_0^{1/3}\left(y_0^{2/3} + z_0^{2/3}\right)$$

$$A = x_0 + x_0^{1/3}\left(y_0^{2/3} + z_0^{2/3}\right) = x_0 + x_0^{1/3}\left(a^{2/3} - x_0^{2/3}\right) = x_0^{1/3}a^{2/3}$$

Similarly, $B = y_0^{1/3}a^{2/3}$ and $C = z_0^{1/3}a^{2/3}$.

$$A^2 + B^2 + C^2 = x_0^{2/3}a^{4/3} + y_0^{2/3}a^{4/3} + z_0^{2/3}a^{4/3} = a^{4/3}\left(x_0^{2/3} + y_0^{2/3} + z_0^{2/3}\right) = a^{4/3}a^{2/3} = a^2$$

28) Find the points on the paraboloid $z = 4x^2 + 9y^2$ at which the normal line is parallel to the line through $P(-2,4,3)$ and $Q(5,-1,2)$.

Solution:

$z = 4x^2 + 9y^2$

$f_x(x,y) = 8x, \quad f_y(x,y) = 18y$

normal vector: $\langle 8x, 18y, -1 \rangle$

$\overrightarrow{PQ} = \langle 5+2, -1-4, 2-3 \rangle = \langle 7, -5, -1 \rangle$

$8x = 7 \Rightarrow x = 7/8$

$18y = -5 \Rightarrow y = -5/18$

$z = 4(7/8)^2 + 9(-5/18)^2 = 541/144$