MATH 267 CHAPTER 16 PARTIAL DIFFERENTIATION

16.6 DIRECTIONAL DERIVATIVES

$f_x(x,y)$ instantaneous rate of change wrt distance in the horizontal direction
$f_y(x,y)$ instantaneous rate of change wrt distance in the vertical direction

To generalize this to the rate of change in *any* direction.

Let $\vec{u} = u_1\hat{i} + u_2\hat{j} = \langle u_1, u_2 \rangle$ be a unit vector. Represent $\vec{u}$ by a vector with initial point $P(x,y)$; then the terminal point will have coordinates $(x+u_1, y+u_2)$. To define the rate of change of $f(x,y)$ with respect to distance in the direction determined by $\vec{u}$.

Consider the line through *P* parallel to $\vec{u}$, and let *Q* be any point on *l*. The vector $\overrightarrow{PQ}$ corresponds to a scalar multiple $s\vec{u} = (su_1)\hat{i} + (su_2)\hat{j} = \langle su_1, su_2 \rangle$ for some *s*. Hence the coordinates of *Q* are $(x+su_1, y+su_2)$ and

$$\left\|\overrightarrow{PQ}\right\| = \left\|s\vec{u}\right\| = |s|\left\|\vec{u}\right\| = |s| .$$

Thus, *s* is the signed distance from *P*, measured along *l*.
If *s* > 0 then $s\vec{u}$ has the same direction as $\vec{u}$.
If *s* < 0 then $s\vec{u}$ has the opposite direction.

If the position of a point changes from *P* to *Q,* then the increment Δw of $w = f(x,y)$ is

$$\Delta w = f(x+su_1, y+su_2) - f(x,y) .$$

The **average rate of change** in $f(x,y)$ is

$$\frac{\Delta w}{s} = \frac{f(x+su_1, y+su_2) - f(x,y)}{s} .$$

To find the **instantaneous rate of change** of $f(x,y)$ at *P* in the direction determined by $\vec{u}$, take the limit…

Definition Let $w = f(x,y)$, and let $\vec{u} = u_1\hat{i} + u_2\hat{j} = \langle u_1, u_2 \rangle$ be a unit vector. The **directional derivative of *f* at** $P(x,y)$ **in the direction of** $\vec{u}$, denoted by $D_{\vec{u}}f(x,y)$, is

$$D_{\vec{u}}f(x,y) = \lim_{s\to 0}\frac{f(x+su_1, y+su_2) - f(x,y)}{s} .$$

Remark If $\vec{a}$ is any vector that has the same direction as $\vec{u}$, then $D_{\vec{u}}f(x,y)$ is also referred to as the **directional derivative in the direction of** $\vec{a}$**.**

Formula for the Directional Derivative

Theorem If *f* is a differentiable function of two variables and $\vec{u} = u_1\hat{i} + u_2\hat{j} = \langle u_1, u_2 \rangle$ is a unit vector, then

$$D_{\vec{u}} f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2 .$$

Proof: Consider *x*, *y*, u_1, and u_2 as fixed (but arbitrary), and let *g* be the function of one variable *s* defined by

$$g(s) = f(x + su_1, y + su_2).$$

Then

$$g'(0) = \lim_{s\to 0}\frac{g(s)-g(0)}{s-0} = \lim_{s\to 0}\frac{f(x+su_1, y+su_2) - f(x,y)}{s} = D_{\vec{u}} f(x,y)$$

Next, consider *g* as a composite function, with

$$w = g(s) = f(r,v) \quad \text{and} \quad r = x + su_1, \quad v = y + su_2 .$$

Thus, *w* is a function of *r* and *v*, and *r* and *v* are functions of one variable *s*. Applying a chain rule,

$$\frac{dw}{ds} = \frac{\partial w}{\partial r}\frac{dr}{ds} + \frac{\partial w}{\partial v}\frac{dv}{ds} .$$

Using the definitions of $w = g(s)$, r, and v, we may rewrite this as

$$g'(s) = f_r(r,v)u_1 + f_v(r,v)u_2 .$$

If we let $s = 0$, then $r = x$, $v = y$, and by the first part of the proof, $g'(0) = D_{\vec{u}} f(x,y)$. Hence

$$D_{\vec{u}} f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2 .$$

Definition Let *f* be a function of two variables. The **gradient** of *f* is the vector function given by

$$\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j} = \langle f_x(x,y),\ f_y(x,y) \rangle .$$

Other notation: grad $f(x,y)$

Exercises Find the gradient of *f* at *P*.

2) $f(x,y) = 7y - 5x; \quad P(2,6)$
Solution:

$$\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j} \text{ or } \langle f_x(x,y),\ f_y(x,y) \rangle = -5\hat{i} + 7\hat{j} \text{ or } \langle -5, 7 \rangle$$

4) $f(x,y) = x\ln(x-y); \quad P(5,4)$
Solution:

$$\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j} = \langle f_x(x,y),\ f_y(x,y) \rangle$$
$$= \left(\ln(x-y) + \frac{x}{x-y}\right)\hat{i} + \left(\frac{-x}{x-y}\right)\hat{j}\Bigg]_{(5,4)} = \left(\ln(5-4) + \frac{5}{5-4}\right)\hat{i} - \frac{5}{5-4}\hat{j} = 5\hat{i} - 5\hat{j} = \langle 5, -5 \rangle$$

6) $f(x,y,z) = xy^2e^z$; $P(2,-1,0)$

Solution:

$$\nabla f(x,y,z) = \langle f_x(x,y,z),\ f_y(x,y,z), f_z(x,y,z)\rangle$$
$$= \langle y^2e^z, 2xye^z, xy^2e^z\rangle\Big]_{(2,-1,0)} = \left\langle (-1)^2 e^0, 2(2)(-1)e^0, (2)(-1)^2 e^0\right\rangle = \langle 1,-4,2\rangle$$

Gradient form of the directional derivative

$$D_{\vec{u}} f(x,y) = \nabla f(x,y)\cdot \vec{u}$$

Exercises Find the directional derivative of *f* at the point *P* in the indicated direction.

8) $f(x,y) = x^3 - 3x^2y - y^3$; $P(1,-2)$, $\vec{u} = \frac{1}{2}\left(-\hat{i} + \sqrt{3}\hat{j}\right)$

Solution:

$$D_{\vec{u}} f(x,y) = \nabla f(x,y)\cdot\vec{u}$$
$$= \langle f_x(x,y), f_y(x,y)\rangle \cdot \left\langle -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right\rangle$$
$$= \langle 3x^2 - 6xy, -3x^2 - 3y^2\rangle \cdot \left\langle -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right\rangle = -\tfrac{1}{2}\left(3x^2 - 6xy\right) + \tfrac{\sqrt{3}}{2}\left(-3x^2 - 3y^2\right)\Big]_{(1,-2)}$$
$$= -\tfrac{1}{2}\left(3(1)^2 - 6(1)(-2)\right) + \tfrac{\sqrt{3}}{2}\left(-3(1)^2 - 3(-2)^2\right) = -\tfrac{1}{2}(3+12) + \tfrac{\sqrt{3}}{2}(-3-12) = -\frac{15+15\sqrt{3}}{2}$$

Gradient Theorem Let *f* be a function of two variables that is differentiable at the point $P(x,y)$.

(i) The maximum value of $D_{\vec{u}} f(x,y)$ at $P(x,y)$ is $\|\nabla f(x,y)\|$.

(ii) The maximum rate of increase of $f(x,y)$ at $P(x,y)$ occurs in the direction of $\nabla f(x,y)$.

Proof:

(i) Consider the point $P(x,y)$ and the vector $\nabla f(x,y)$ as fixed (but arbitrary) and the unit vector $\vec{u}$ as a variable. Let γ be the angle between $\vec{u}$ and $\nabla f(x,y)$.

$$D_{\vec{u}} f(x,y) = \nabla f(x,y)\cdot\vec{u}$$
$$= \|\nabla f(x,y)\|\|\vec{u}\|\cos\gamma$$
$$= \|\nabla f(x,y)\|\cos\gamma$$

Since $-1 \le \cos\gamma \le 1$, the maximum value of $D_{\vec{u}} f(x,y)$ occurs if $\cos\gamma = 1$ or when $D_{\vec{u}} f(x,y) = \|\nabla f(x,y)\|$.

(ii) $D_{\vec{u}} f(x,y)$ is the rate of change of $f(x,y)$ wrt distance at $P(x,y)$ in the direction of $\vec{u}$. From (i) $D_{\vec{u}} f(x,y)$ has its max value if $\cos\gamma = 1$ or if $\gamma = 0$ which means that $\vec{u}$ has the same direction as $\nabla f(x,y)$.

Geometric Interpretation of the Gradient Theorem

Let *S* denote the graph of $z = f(x,y)$ in an *xyz*-coordinate system. The point $P(x,y)$ and the vector $\nabla f(x,y)$ are in the *xy*-plane, and $Q(x,y,f(x,y))$ is the point on *S* corresponding to *P*. If a (variable) point in the xy-plane moves through *P* in the direction of $\nabla f(x,y)$, then by part (ii) of the

Gradient Theorem, the corresponding point on *S* moves up a curve *C* of ***steepest ascent*** at the point *Q* on the graph of *f*.

Corollary Let *f* be a function of two variables that is differentiable at the point $P(x,y)$.

(i) The minimum value of $D_{\vec{u}} f(x,y)$ at $P(x,y)$ is $-\|\nabla f(x,y)\|$.

(ii) The minimum rate of increase (or maximum rate of decrease) of $f(x,y)$ at $P(x,y)$ occurs in the direction of $-\nabla f(x,y)$.

Exercise

(a) Find the directional derivative of *f* at *P* in the direction from *P* to *Q*.

(b) Find a unit vector in the direction in which *f* increases most rapidly at *P*, and find the rate of change of *f* in that direction.

(c) Find a unit vector in the direction in which *f* decreases most rapidly at *P*, and find the rate of change of *f* in that direction.

22) $f(x,y) = \sin(2x - y); \quad P(-\pi/3, \pi/6), \quad Q(0,0)$

Solution:

a) $f(x,y) = \sin(2x - y); \quad P(-\pi/3, \pi/6), \quad Q(0,0)$

$$\overrightarrow{PQ} = \langle \pi/3, -\pi/6 \rangle \Rightarrow \|\overrightarrow{PQ}\| = \sqrt{\frac{\pi^2}{9} + \frac{\pi^2}{36}} = \frac{\pi}{6}\sqrt{5};\ \vec{u} = \frac{6}{\pi\sqrt{5}}\left\langle \frac{\pi}{3}, -\frac{\pi}{6} \right\rangle = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u} = \langle 2\cos(2x - y), -\cos(2x - y) \rangle \cdot \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

$$= \frac{2}{\sqrt{5}} 2\cos(2x - y) + \frac{1}{\sqrt{5}}\cos(2x - y)\Big]_{(-\pi/3,\pi/6)} = \frac{4}{\sqrt{5}}\cos\left(-\frac{2\pi}{3} - \frac{\pi}{6}\right) + \frac{1}{\sqrt{5}}\cos\left(-\frac{2\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{4}{\sqrt{5}}\cos\left(-\frac{5\pi}{6}\right) + \frac{1}{\sqrt{5}}\cos\left(-\frac{5\pi}{6}\right) = \frac{4}{\sqrt{5}}\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{\sqrt{5}}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{2\sqrt{3}}{\sqrt{5}} - \frac{\sqrt{3}}{2\sqrt{5}} = \frac{-5\sqrt{3}}{2\sqrt{5}} = -\frac{\sqrt{15}}{2}$$

b) $$\nabla f\left(-\frac{\pi}{3}, \frac{\pi}{6}\right) = \left\langle 2\cos\left(-\frac{5\pi}{6}\right), -\cos\left(-\frac{5\pi}{6}\right) \right\rangle = \left\langle -\sqrt{3}, \frac{\sqrt{3}}{2} \right\rangle$$

$$\left\|\nabla f\left(-\frac{\pi}{3}, \frac{\pi}{6}\right)\right\| = \sqrt{3 + \frac{3}{4}} = \frac{\sqrt{15}}{2}$$

$$\vec{v} = \left\langle -\frac{2\sqrt{3}}{\sqrt{15}}, \frac{\sqrt{3}}{\sqrt{15}} \right\rangle = \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

c) $$-\vec{v} = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle; \qquad -\left\|\nabla f\left(-\frac{\pi}{3}, \frac{\pi}{6}\right)\right\| = -\frac{\sqrt{15}}{2}$$

Directional Derivative for a Function of Three Variables

The directional derivative for a function of three variables is defined similarly.

Definition If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ is a unit vector, then

$$D_{\vec{u}} f(x,y,z) = \lim_{s \to 0} \frac{f(x + su_1, y + su_2, z + su_3)}{s}.$$

Corollary $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$

Theorem If *f* is a differentiable function of three variables and $\vec{u} = \langle u_1, u_2, u_3 \rangle$ is a unit vector, then

$$\begin{aligned} D_{\vec{u}} f(x,y,z) &= \nabla f(x,y,z) \cdot \vec{u} \\ &= f_x(x,y,z)u_1 + f_y(x,y,z)u_2 + f_z(x,y,z)u_3 \end{aligned}$$

Exercise

18) $f(x,y,z) = \sqrt{xy}\sin z; \quad P(4,9,\pi/4), \quad \vec{a} = 2\hat{i} + 3\hat{j} - 2k$

Solution:

$$\|\vec{a}\| = \sqrt{2^2 + 3^2 + 2^2} = \sqrt{17} \Rightarrow \vec{u} = \frac{2\hat{i} + 3\hat{j} - 2k}{\sqrt{17}}$$

$$\begin{aligned} D_{\vec{u}} f(x,y,z) &= \nabla f(x,y,z) \cdot \vec{u} \\ &= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle \cdot \frac{1}{\sqrt{17}} \langle 2,3,-2 \rangle \\ &= \left\langle \frac{y}{2\sqrt{xy}} \sin z, \frac{x}{2\sqrt{xy}} \sin z, \sqrt{xy}\cos z \right\rangle \cdot \frac{1}{\sqrt{17}} \langle 2,3,-2 \rangle \\ &= \frac{2}{\sqrt{17}} \frac{y}{\sqrt{xy}} \sin z + \frac{3}{\sqrt{17}} \frac{x}{\sqrt{xy}} \sin z - \frac{2}{\sqrt{17}} \sqrt{xy}\cos z \Big]_{(4,9,\pi/4)} \\ &= \frac{2}{\sqrt{17}} \frac{9}{\sqrt{4\cdot 9}} \sin\frac{\pi}{4} + \frac{3}{\sqrt{17}} \frac{4}{\sqrt{4\cdot 9}} \sin\frac{\pi}{4} - \frac{2}{\sqrt{17}} \sqrt{4\cdot 9}\cos\frac{\pi}{4} \\ &= \frac{2}{\sqrt{17}} \frac{9}{6} \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{17}} \frac{4}{6} \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{17}} 12 \cdot \frac{1}{\sqrt{2}} = \frac{3 + 2 - 24}{\sqrt{34}} = \frac{-19}{\sqrt{34}} \end{aligned}$$

26) The surface of a lake is represented by a region *D* in the *xy*-plane such that the depth (in feet) under the point (x,y) is $f(x,y) = 300 - 2x^2 - 3y^2$.

(a) In what direction should a boat at $P(4,9)$ sail in order for the depth of the water to decrease most rapidly?

(b) In what direction does the depth remain the same?

Solution:

(a) $f(x,y) = 300 - 2x^2 - 3y^2$

$$\nabla f(x,y) = \langle -4x, -6y \rangle$$

$$-\nabla f(4,9) = -\langle -16, -54 \rangle = \langle 16, 54 \rangle$$

(b) The depth remains the same if its rate of change is 0.

$$\begin{aligned} \nabla f(4,9) \cdot \vec{w} = 0 &\Rightarrow \langle -16, -54 \rangle \cdot \langle w_1, w_2 \rangle = 0 \\ &\Rightarrow -16w_1 - 54w_2 = 0 \\ &\Rightarrow w_1 = -\frac{27}{8} w_2 \end{aligned}$$

Thus, if $w_2 = -8$, then $w_1 = 27$, and the depth remains the same in the direction $\vec{w}$.