# MATH 267 CHAPTER 16 PARTIAL DIFFERENTIATION

## 16.5 CHAIN RULES

**Theorem** If $w = f(u,v)$ , with $u = g(x,y), v = h(x,y),$ and if *f, g,* and *h* are differentiable, then

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

Proof: If *x* is given an increment $\Delta x$ and *y* is held constant $(\Delta y = 0)$ we get

$$\begin{aligned}\Delta u &= g(x+\Delta x, y) - g(x,y)\\ \Delta v &= h(x+\Delta x, y) - h(x,y)\end{aligned} \qquad (1)$$

And these lead to

$$\Delta w = f(u+\Delta u, v+\Delta v) - f(u,v)$$

Since *f* is differentiable, we have,

$$\Delta w = \frac{\partial w}{\partial u}\Delta u + \frac{\partial w}{\partial v}\Delta v + \in_1 \Delta u + \in_2 \Delta v \qquad (2)$$

where $\in_1$ and $\in_2$ are functions of $\Delta u$ and $\Delta v$ that have the limit 0 as $(\Delta u, \Delta v) \to (0,0)$. Moreover, we may assume that both $\in_1$ and $\in_2$ equal 0 if $(\Delta u, \Delta v) = (0,0)$, otherwise we can always choose other functions that satisfy this condition and behaves like $\in_1$ and $\in_2$ everywhere else. Thus, the functions $\in_1$ and $\in_2$ in (2) are continuous at $(0,0)$. Dividing both sides of (2) by $\Delta x$ gives

$$\frac{\Delta w}{\Delta x} = \frac{\partial w}{\partial u}\frac{\Delta u}{\Delta x} + \frac{\partial w}{\partial v}\frac{\Delta v}{\Delta x} + \in_1 \frac{\Delta u}{\Delta x} + \in_2 \frac{\Delta v}{\Delta x} \qquad (3)$$

If we regard *w* as a function of *x* and *y*, then

$$\lim_{\Delta x \to 0}\frac{\Delta w}{\Delta x} = \frac{\partial w}{\partial x}.$$

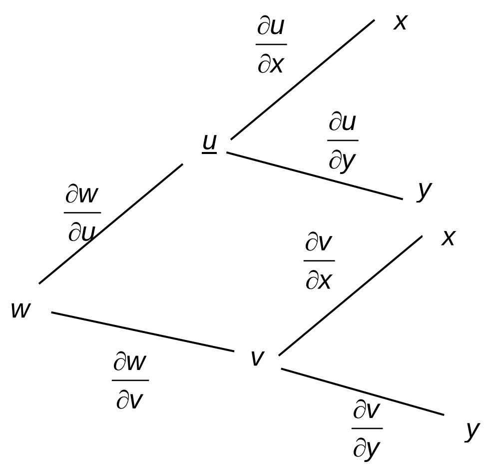
Also, from (1),

$$\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \quad \text{and} \quad \lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} = \frac{\partial v}{\partial x}.$$

If $\Delta x \to 0$, we see from (1) that $\Delta u \to 0$ and $\Delta v \to 0$ and thus so do $\in_1$ and $\in_2$. Taking the limit in (3) as $\Delta x \to 0$, we obtain

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}.$$

**Tree Diagram**




We can apply chain rules to composite functions of any number of variables and construct tree diagrams to help formulate these rules.

For instance, if *w* is a function of *u*, *v,* and *r,* and *u*, *v,* and *r* are each functions of *x*, *y*, and *z,* then

$$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial r}\frac{\partial r}{\partial y}.$$

**Exercises**

**2)** Find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$: $w=uv+v^2;\quad u=x\sin y,\quad v=y\sin x$

Solution:

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$
$$=v\cdot\sin y+(u+2v)y\cos x$$
$$=y\sin x\sin y+y(x\sin y+2y\sin x)\cos x$$

$$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$
$$=v\cdot x\cos y+(u+2v)\sin x$$
$$=xy\sin x\cos y+(x\sin y+2y\sin x)\sin x$$

**4)** Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$: $w=e^{tv};\quad t=r+s,\quad v=rs$

Solution:

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial t}\frac{\partial t}{\partial r}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial r}$$
$$=ve^{tv}\cdot 1+te^{tv}\cdot s$$
$$=e^{tv}(v+ts)$$
$$=se^{rs(r+s)}(2r+s)$$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial t}\frac{\partial t}{\partial s}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$
$$=e^{tv}(v+tr)$$
$$=re^{rs(r+s)}(r+2s)$$

**8)** Find $\frac{\partial r}{\partial u}$, $\frac{\partial r}{\partial v}$, and $\frac{\partial r}{\partial t}$. $\quad r = w^2 \cos z; \quad w = u^2 vt, \quad z = ut^2$

Solution:

$$\begin{aligned}\frac{\partial r}{\partial u} &= \frac{\partial r}{\partial w}\frac{\partial w}{\partial u} + \frac{\partial r}{\partial z}\frac{\delta z}{\delta u}\\ &= (2w\cos z)(2uvt) + (-w^2 \sin z)(t^2)\\ &= 4uvwt\cos z - w^2t^2 \sin z\\ &= 4uv(u^2vt)t\cos(ut^2) - (u^2vt)^2 t^2 \sin(ut^2)\\ &= 4u^3v^2t^2\cos(ut^2) - u^4v^2t^4\sin(ut^2)\end{aligned}$$

$$\begin{aligned}\frac{\partial r}{\partial v} &= \frac{\partial r}{\partial w}\frac{\partial w}{\partial v} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial v}\\ &= (2w\cos z)(u^2t) + (-w^2\sin z)(0)\\ &= 2(u^2vt)u^2t\cos(ut^2)\\ &= 2u^4vt^2\cos(ut^2)\end{aligned}$$

$$\begin{aligned}\frac{\partial r}{\partial t} &= \frac{\partial r}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial t}\\ &= (2w\cos z)(u^2v) + (-w^2\sin z)(2ut)\\ &= 2(u^2vt)(u^2v)\cos(ut^2) - 2ut(u^2vt)^2\sin(ut^2)\\ &= 2u^4v^2t\cos(ut^2) - 2u^5v^2t^3\sin(ut^2)\end{aligned}$$

**10)** Find $\frac{\partial s}{\partial y}$ if $s = tr + ue^v$, where $t = xy^2z$, $r = x^2yz$, $u = xyz^2$, and $v = xyz$.

Solution:

$$\begin{aligned}\frac{\partial s}{\partial y} &= \frac{\partial s}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial s}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial s}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial y}\\ &= r\cdot 2xyz + t\cdot x^2z + e^v\cdot xz^2 + ue^v\cdot xz\\ &= x^2yz\cdot 2xyz + xy^2z\cdot x^2z + xz^2e^{xyz} + xyz^2e^{xyz}\cdot xz\\ &= 2x^3y^2z^2 + x^3y^2z^2 + xz^2e^{xyz} + x^2yz^3e^{xyz}\\ &= 3x^3y^2z^2 + xz^2e^{xyz} + x^2yz^3e^{xyz}\end{aligned}$$

If *w* is a function of several variables, each of which is a function of one variable, say *t*, then *w* is a function of the one variable *t* and we may consider $\frac{dw}{dt}$.

**12)** $w = \ln(u+v); \quad u = e^{-2t}, \quad v = t^3 - t^2$

Solution:

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial v}\frac{dv}{dt}\\ &= \frac{1}{u+v}(-2e^{-2t}) + \frac{1}{u+v}(3t^2 - 2t)\\ &= \frac{-2e^{-2t} + 3t^2 - 2t}{e^{-2t} + t^3 - t^2}\end{aligned}$$

**Implicit Partial Differentiation**

**Theorem** If an equation $F(x,y) = 0$ determines, implicitly, a differentiable function *f* of one variable *x* such that $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}.$$

Proof: Let $w=F(u,y)$, where $u=x$ and $y=f(x)$. Then

$$\frac{dw}{dx}=\frac{\partial w}{\partial u}\frac{du}{dx}+\frac{\partial w}{\partial y}\frac{dy}{dx}. \qquad (*)$$

Since $w=F(x,f(x))=0$ for every $x$, it follows that $\frac{dw}{dx}=0$. Also, since $u=x$ and $y=f(x)$,

$$\frac{du}{dx}=1 \quad \text{and} \quad \frac{dy}{dx}=f'(x).$$

Substituting in (*) we get

$$0=\frac{\partial w}{\partial u}(1)+\frac{\partial w}{\partial y}f'(x).$$

If $\frac{\partial w}{\partial y}\neq 0$, then since $u=x$,

$$f'(x)=-\frac{\partial w/\partial u}{\partial w/\partial y}=-\frac{\partial w/\partial x}{\partial w/\partial y}=-\frac{F_x(x,y)}{F_y(x,y)}.$$

**Exercise**

**16)** $x^4+2x^2y^2-3xy^3+2x=0$

Solution:

$$F(x,y)=x^4+2x^2y^2-3xy^3+2x$$
$$F_x(x,y)=4x^3+4xy^2-3y^3+2$$
$$F_y(x,y)=4x^2y-9xy^2$$
$$\frac{dy}{dx}=-\frac{F_x(x,y)}{F_y(x,y)}=-\frac{4x^3+4xy^2-3y^3+2}{4x^2y-9xy^2}$$

**Theorem** If an equation $F(x,y,z)=0$ determines an implicit differentiable function $f$ of two variables $x$ and $y$ such that $z=f(x,y)$ for every $(x,y)$ in the domain of $f$, then

$$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)}, \qquad \frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)}.$$

**20)** $xz^2+2x^2y-4y^2z+3y-2=0$

Solution:

$$F(x,y,z)=xz^2+2x^2y-4y^2z+3y-2$$
$$F_x(x,y,z)=z^2+4xy$$
$$F_y(x,y,z)=2x^2-8yz+3$$
$$F_z(x,y,z)=2xz-4y^2$$
$$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)}=-\frac{z^2+4xy}{2xz-4y^2}$$
$$\frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)}=-\frac{2x^2-8yz+3}{2xz-4y^2}$$

**24)** The equal sides and the included angle of an isosceles triangle are increasing at rates of 0.1 ft/hr and $2^\circ/\text{hr}$, resp. Find the rate at which the area of the triangle is increasing at the time when the length of each of the equal sides is 20 ft and the included angle is $60^\circ$.

Solution:

$$A=\frac{1}{2}s^2\sin\theta$$

$$\frac{dA}{dt}=\frac{\partial A}{\partial s}\frac{ds}{dt}+\frac{\partial A}{\partial\theta}\frac{d\theta}{dt}=s\sin\theta\frac{ds}{dt}+\frac{1}{2}s^2\cos\theta\frac{d\theta}{dt}$$

Substitute $s=20,\ \theta=60^\circ,\ \dfrac{ds}{dt}=0.1,\ \dfrac{d\theta}{dt}=2^\circ=\dfrac{\pi}{90}$ :

$$\frac{dA}{dt}=20\left(\sin 60^\circ\right)(0.1)+\frac{1}{2}(20)^2\left(\cos 60^\circ\right)\left(\frac{\pi}{90}\right)=20\cdot\frac{\sqrt{3}}{2}(0.1)+200\cdot\frac{1}{2}\cdot\frac{\pi}{90}\approx 5.22\text{ ft}^2/\text{hr}$$

**26)** If the base radius *r* and altitude *h* of a right circular cylinder are changing at the rates $\dfrac{dr}{dt}$ and $\dfrac{dh}{dt}$, resp., find a formula for $\dfrac{dV}{dt}$, where *V* is the volume of the cylinder.

Solution:

$$V=\pi r^2h$$

$$\frac{dV}{dt}=\frac{\partial V}{\partial r}\frac{dr}{dt}+\frac{\partial V}{\partial h}\frac{dh}{dt}=2\pi rh\frac{dr}{dt}+\pi r^2\frac{dh}{dt}$$

**28)** Sand is leaking out of a hole in a container at a rate of $6\text{ in}^3/\text{min}$. As it leaks out it forms a pile in the shape of a right circular cone whose base radius is increasing at the rate of $1/4$ in/min. If, at the instant that 40 $\text{in}^3$ has leaked out, the radius is 5 inches, find the rate at which the height of the pile is increasing.

Solution:

$$V=\frac{1}{3}\pi r^2h\Rightarrow h=\frac{3V}{\pi r^2}$$

$$\frac{dh}{dt}=\frac{\partial h}{\partial V}\frac{dV}{dt}+\frac{\partial h}{\partial r}\frac{dr}{dt}=\frac{3}{\pi r^2}\frac{dV}{dt}-\frac{6V}{\pi r^3}\frac{dr}{dt}$$

Substitute $V=40,\ \dfrac{dV}{dt}=6,\ \dfrac{dr}{dt}=\dfrac{1}{4},\ r=5$:

$$\frac{dh}{dt}=\frac{3}{\pi(5)^2}(6)-\frac{6(40)}{\pi(5)^3}\left(\frac{1}{4}\right)=\frac{18}{25\pi}-\frac{60}{125\pi}\approx 0.076\text{ in/min}$$

**32)** A function *f* of two variables is ***homogeneous of degree n*** if $f(tx,ty)=t^nf(x,y)$ for every *t* such that $(tx,ty)$ is in the domain of of *f*. Show that, for such functions,

$$xf_x(x,y)+yf_y(x,y)=nf(x,y).$$

(Hint: Differentiate $f(tx,ty)$ wrt *t*.)

Solution:

$$f(tx,ty)=t^nf(x,y)$$

Let $u=tx,\quad v=ty$

$$f(u,v)=f(tx,ty)$$

$$\frac{df}{dt}=\frac{\partial f}{\partial u}\frac{du}{dt}+\frac{\partial f}{\partial v}\frac{dv}{dt}$$

$$=xf_u(u,v)+yf_v(u,v)$$

Also, $f(u,v)=t^nf(x,y)$

$$\frac{\partial f}{\partial t}=nt^{n-1}f(x,y)$$

Since *t* is any real number, let $t=1$:

$$xf_x(x,y)+yf_y(x,y)=nf(x,y)$$

Find the degree of the homogeneous function *f* and verify the formula

$$xf_x(x,y)+yf_y(x,y)=nf(x,y)$$

**34)** $f(x,y)=\dfrac{x^3y}{x^2+y^2}$

Solution:

$$f(tx,ty)=\frac{(tx)^3(ty)}{(tx)^2+(ty)^2}=\frac{t^4x^3y}{t^2\left(x^2+y^2\right)}=t^2\,\frac{x^3y}{x^2+y^2}=t^2f(x,y)\Rightarrow \text{degree } 2$$

$$xf_x(x,y)+yf_y(x,y)=x\left[\frac{\left(x^2+y^2\right)3x^2y-x^3y(2x)}{\left(x^2+y^2\right)^2}\right]+y\left[\frac{\left(x^2+y^2\right)x^3-x^3y(2y)}{\left(x^2+y^2\right)^2}\right]$$

$$=x\left[\frac{3x^4y+3x^2y^3-2x^4y}{\left(x^2+y^2\right)^2}\right]+y\left[\frac{x^5+x^3y^2-2x^3y^2}{\left(x^2+y^2\right)^2}\right]$$

$$=\frac{x^5y+3x^3y^3+x^5y-x^3y^3}{\left(x^2+y^2\right)^2}=\frac{2x^5y+2x^3y^3}{\left(x^2+y^2\right)^2}=\frac{2x^3y\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}$$

$$=2\frac{x^3y}{x^2+y^2}=2f(x,y)$$