MATH 267 CHAPTER 16 PARTIAL DIFFERENTIATION

16.4 INCREMENTS AND DIFFERENTIALS

Definition Let $w = f(x,y)$ and let Δx and Δy be increments of x and y, respectively. The **increment** Δw of $w = f(x,y)$ is

$$\Delta w = f(x+\Delta x, y+\Delta y) - f(x,y).$$

Δw represents the change in the function value if (x,y) changes to $(x+\Delta x, y+\Delta y)$.

Example Let $w = f(x,y) = xy - y^2$.

- **a)** Find Δw.
- **b)** Use Δw to calculate the change in $f(x,y)$ if (x,y) changes from $(2,1)$ to $(1.99,1.02)$.

Solution:

a)
$$\begin{aligned}\Delta w &= f(x+\Delta x, y+\Delta y) - f(x,y)\\ &= (x+\Delta x)(y+\Delta y) - (y+\Delta y)^2 - xy + y^2\\ &= xy + x\Delta y + y\Delta x + (\Delta x)(\Delta y) - y^2 - 2y\Delta y - (\Delta y)^2 - xy + y^2\end{aligned}$$
$$\Delta w = x\Delta y + y\Delta x + (\Delta x)(\Delta y) - 2y\Delta y - (\Delta y)^2$$

b) Use $x = 2,\ y = 1,\ \Delta x = -0.01,\ \Delta y = 0.02$
$$\begin{aligned}\Delta w &= 2(0.02) + 1(-0.01) + (0.02)(-0.01) - 2(1)(0.02) - (0.02)^2\\ &= 0.04 - 0.01 - 0.0002 - 0.04 - 0.0004\\ &= -0.0106\end{aligned}$$

Note: The same answer will be obtained if we calculate $f(1.99,1.02) - f(2,1)$

Theorem Let $w = f(x,y)$, where the function f is defined on a rectangular region $R = \{(x,y) : a<x<b,\ c<y<d\}$. Suppose f_x and f_y exist throughout R and are continuous at the point (x_0,y_0) in R. If $(x_0+\Delta x, y_0+\Delta y)$ is in R and

$$\Delta w = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0),$$

then

$$\Delta w = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy that have the limit 0 as $(\Delta x, \Delta y) \to (0,0)$.

Proof:
$$\begin{aligned}\Delta w &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)\\ &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y) + [f(x_0, y_0+\Delta y) - f(x_0,y_0)] \quad (1)\end{aligned}$$

Consider Δy as a constant and define a function g of *one* variable x by letting

$$g(x) = f(x, y_0+\Delta y) \quad \text{for} \quad a<x<b.$$

Then

$$g'(x) = f_x(x, y_0+\Delta y) \quad \text{for} \quad a<x<b.$$

Applying the MVT to g on the interval $[x_0, x_0 + \Delta x]$, we get

$$g(x_0 + \Delta x) - g(x_0) = g'(u)\Delta x\,,$$

where u is between x_0 and $x_0 + \Delta x$. Since $g(x) = f(x, y_0 + \Delta y)$ and $g'(x) = f_x(x, y_0 + \Delta y)$, the last equation can be written as

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) = f_x(u, y_0 + \Delta y)\Delta x \qquad (2)\,.$$

Similarly, if we let $h(y) = f(x_0, y)$ for $c < y < d$, then $h'(y) = f_y(x_0, y)$.
Applying the MVT to the function h on the interval $[y_0, y_0 + \Delta y]$, we get

$$h(y_0 + \Delta y) - h(y_0) = h'(v)\Delta y$$

where v is between y_0 and $y_0 + \Delta y$, or, equivalently,

$$f(x_0, y_0 + \Delta y) - f(x_0, y_0) = f_y(x_0, v)\Delta y \qquad (3)$$

Substituting (2) and (3) into the expression for Δw in (1) gives us

$$\Delta w = f_x(u, y_0 + \Delta y)\Delta x + f_y(x_0, v)\Delta y \qquad (4)$$

Define ϵ_1 and ϵ_2 as follows:

$$\epsilon_1 = f_x(u, y_0 + \Delta y) - f_x(x_0, y_0)$$
$$\epsilon_2 = f_y(x_0, v) - f_y(x_0, y_0)$$

Using the fact that f_x and f_y are continuous and noting that

$u \to x_0$ as $\Delta x \to 0$ and $v \to y_0$ as $\Delta y \to 0$ then
$\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Write the equations for ϵ_1 and ϵ_2 in the form

$$f_x(u, y_0 + \Delta y) = f_x(x_0, y_0) + \epsilon_1$$
$$f_y(x_0, v) = f_y(x_0, y_0) + \epsilon_2$$

and then substitute into (4) and get

$$\Delta w = [f_x(x_0, y_0) + \epsilon_1]\Delta x + [f_y(x_0, y_0) + \epsilon_2]\Delta y$$

and we get the conclusion of the theorem.

Example Let $w = f(x, y) = xy - y^2$. Find expressions for ϵ_1 and ϵ_2 that satisfy the conclusion of the theorem with $(x_0, y_0) = (x, y)$.

Solution: From the first example, we have

$$\Delta w = x\Delta y + y\Delta x + (\Delta x)(\Delta y) - 2y\Delta y - (\Delta y)^2$$

or, equivalently,

$$\Delta w = y\Delta x + (x - 2y)\Delta y + (\Delta y)(\Delta x) - (\Delta y)(\Delta y)$$

This is of the form given in the theorem with

$$f_x(x,y) = y, \quad f_y(x,y) = x - 2y, \quad \epsilon_1 = \Delta y, \quad \epsilon_2 = \Delta y \ .$$

Definition Let $w = f(x,y)$, and let Δx and Δy be increments of x and y, respectively.

(i) The **<u>differentials</u>** dx and dy of the independent variables x and y are

$$dx = \Delta x \text{ and } dy = \Delta y \ .$$

(ii) The **<u>differential</u>** dw of the dependent variable w is

$$dw = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy \ .$$

Note. If f satisfies the hypotheses of the theorem, then using the conclusion of the theorem with (x_0, y_0) replaced by (x,y), we get

$$\Delta w = dw + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

and hence

$$\Delta w - dw = \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where both ϵ_1 and ϵ_2 approach 0 as $(\Delta x, \Delta y) \to (0,0)$. It follows that if Δx and Δy are small, then $\Delta w - dw \approx 0$, i.e. $dw \approx \Delta w$.

Exercise Find Δw, dw, and $dw - \Delta w$.

2) $w = xy - y^2 + 3x$

Solution:

$$\begin{aligned}
\Delta w &= (x+\Delta x)(y+\Delta y) - (y+\Delta y)^2 + 3(x+\Delta x) - xy + y^2 - 3x \\
&= xy + x\Delta y + y\Delta x + (\Delta x)(\Delta y) - y^2 - 2y\Delta y - (\Delta y)^2 + 3x + 3\Delta x - xy + y^2 - 3x \\
&= x\Delta y + y\Delta x + (\Delta x)(\Delta y) - 2y\Delta y - (\Delta y)^2 + 3\Delta x \\
&= (y+3)\Delta x + (x-2y)\Delta y + (\Delta x)(\Delta y) - (\Delta y)^2
\end{aligned}$$

$$\begin{aligned}
dw &= \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy \\
&= (y+3)dx + (x-2y)dy
\end{aligned}$$

$$dw - \Delta w = (\Delta x)(\Delta y) - (\Delta y)^2$$

Use differentials to approximate the change in f.

20) $f(x,y) = x^2 - 2xy + 3y; \quad (1,2)$ to $(1.03, 1.99)$

Solution:

$$f(x,y) = x^2 - 2xy + 3y; \quad (1,2) \text{ to } (1.03, 1.99)$$

$$dx = 0.03, \quad dy = -0.01$$

$$f_x(x,y) = 2x - 2y\Big|_{(1,2)} = 2(1) - 2(2) = -2$$

$$f_y(x,y) = -2x + 3\Big|_{(1,2)} = -2(1) + 3 = 1$$

$$df = f_x(x,y)dx + f_y(,y)dy$$

At the given values,

$$df = (-2)(0.03) + (1)(-0.01) = -0.06 - 0.01 = -0.07$$

24) The two shortest sides of a right triangle are measured as 3 cm and 4 cm, resp., with a maximum possible error of ± 0.02 cm in each measurement. Use differentials to approximate the maximum error in the calculated value of **a)** the hypotenuse, and **b)** the area of the triangle.
Solution:

a) $x = 3, \; y = 4, \quad dx = dy = \pm 0.02$

$$z^2 = x^2 + y^2 \Leftrightarrow z = \left(x^2 + y^2\right)^{1/2}$$

$$z_x = \left.\frac{x}{\left(x^2+y^2\right)^{1/2}}\right]_{(3,4)} = \frac{3}{5} \qquad z_y = \left.\frac{y}{\left(x^2+y^2\right)^{1/2}}\right]_{(3,4)} = \frac{4}{5}$$

$$dz = z_x dx + z_y dy$$

At the given values, $dz = \frac{3}{5}(\pm 0.02) + \frac{4}{5}(\pm 0.02) = \pm 0.028$ cm.

b) $A(x,y) = \frac{1}{2}xy$

$$A_x(x,y) = \left.\frac{1}{2}y\right]_4 = 2 \qquad A_y(x,y) = \left.\frac{1}{2}x\right]_3 = \frac{3}{2}$$

$$dA = A_x(x,y)dx + A_y(x,y)dy$$

At the given values, $dA = (2)(\pm 0.02) + \left(\frac{3}{2}\right)(\pm 0.02) = \pm 0.07 \text{ cm}^2$.

Recall: If a measurement w has an error approximated by dw, then the

$$\underline{\textbf{average error}} \approx \frac{dw}{w}$$

and

$$\underline{\textbf{percentage error}} = \frac{dw}{w} \cdot 100\%$$

Exercise

30) Suppose that when the ideal gas law $PV = kT$ is used, there are percentage errors of $\pm 0.8\%$ and $\pm 0.5\%$ in the measurements of T and P, resp. Approximate the maximum percentage error in the calculated value of V.
Solution:

We are given $\frac{dT}{T} = \pm 0.008$ and $\frac{dP}{P} = \pm 0.005$.

$$PV = kT \Rightarrow V = \frac{kT}{P}$$

$$dV = \left(\frac{\partial}{\partial T}\frac{kT}{P}\right)dT + \left(\frac{\partial}{\partial P}\frac{kT}{P}\right)dP$$

$$= \frac{k}{P}dT - \frac{kT}{P^2}dP = \frac{kT}{P}\frac{dT}{T} - \frac{kT}{P}\frac{dP}{P} \quad = \frac{kT}{P}\left(\frac{dT}{T} - \frac{dP}{P}\right)$$

$$dV = V\left(\frac{dT}{T} - \frac{dP}{P}\right)$$

$$\frac{dV}{V} = \frac{dT}{T} - \frac{dP}{P} = \pm 0.008 \pm 0.005$$

The maximum % error is $\pm 1.3\%$.

Definition Let $w = f(x,y)$. The function *f* is **differentiable** at (x_0,y_0) if Δw can be expressed in the form

$$\Delta w = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \in_1 \Delta x + \in_2 \Delta y ,$$

where $\in_1$ and $\in_2$ are functions of Δx and Δy that have the limit 0 as $(\Delta x,\Delta y)\to(0,0)$.

A function *f* of two variables is said to be **differentiable on a region** *R* if it is differentiable at every point of *R.*

Exercise Find expressions for $\in_1$ and $\in_2$ that satisfy the definition.

4) $f(x,y)=(2x-y)^2$

Solution:

$$f_x(x,y)=4(2x-y) \qquad f_y(x,y)=-2(2x-y)$$

$$\begin{aligned}\Delta f &= f(x+\Delta x, y+\Delta y)-f(x,y)\\ &=\left[2(x+\Delta x)-(y+\Delta y)\right]^2-(2x-y)^2\\ &=4(x+\Delta x)^2-4(x+\Delta x)(y+\Delta y)+(y+\Delta y)^2-(2x-y)^2\\ &=4\left(x^2+2x\Delta x+\Delta^2 x\right)-4\left(xy+x\Delta y+y\Delta x+\Delta x\Delta y\right)+y^2+2y\Delta y+\Delta^2 y-4x^2+4xy-y^2\\ &=8x\Delta x+4\Delta^2 x-4x\Delta y-4y\Delta x-4\Delta x\Delta y+2y\Delta y+\Delta^2 y\\ &=8x\Delta x-4y\Delta x-4x\Delta y+2y\Delta y+4\Delta^2 x-4\Delta x\Delta y+\Delta^2 y\\ &=4(2x-y)\Delta x-2(2x-y)\Delta y+4\Delta^2 x-4\Delta x\Delta y+\Delta^2 y\end{aligned}$$

By definition:

$$\begin{aligned}\Delta f &= f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y\\ &=4(2x-y)\Delta x-2(2x-y)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y\end{aligned}$$

Comparing, we see that

$$\in_1 \Delta x+\in_2 \Delta y = 4\Delta^2 x-4\Delta x\Delta y+\Delta^2 y=(4\Delta x-4\Delta y)\Delta x+(\Delta y)\Delta y$$

$$\Rightarrow \in_1 = 4\Delta x-4\Delta y \text{ and } \in_2=\Delta y$$

Theorem If $w = f(x,y)$ and if f_x and f_y are continuous on a rectangular region *R,* then *f* is differentiable on *R.*

Theorem If a function *f* of two variables is differentiable at (x_0,y_0), then *f* is continuous at (x_0,y_0).

Proof: $\Delta w = f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)$

$$\Delta w=\left[f_x(x_0,y_0)+\in_1\right]\Delta x+\left[f_y(x_0,y_0)+\in_2\right]\Delta y$$

Equate these expressions and let $x = x_0+\Delta x,\ \ y=y_0+\Delta y$, then

$$f(x,y)-f(x_0,y_0)=\left[f_x(x_0,y_0)+\in_1\right](x-x_0)+\left[f_y(x_0,y_0)+\in_2\right](y-y_0).$$

It follows that

$$\lim_{(x,y)\to(x_0,y_0)}\left[f(x,y)-f(x_0,y_0)\right]=0$$

or

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0).$$

Corollary If *f* is a function of two variables and if f_x and f_y are continuous on a rectangular region *R*, then *f* is continuous on *R*.

Extending to a function of three variables:

Definition Let $w = f(x,y,z)$ and let $\Delta x, \Delta y,$ and Δz be increments of *x, y,* and *z*, respectively.

(i) The **differentials** *dx, dy,* and *dz* of the independent variables *x, y,* and *z* are

$$dx = \Delta x, \quad dy = \Delta y, \quad dz = \Delta z .$$

(ii) The **differential** *dw* of the dependent variable *w* is

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz .$$

Exercise

22) $f(x,y,z) = xy + xz + yz;\ (-1,2,3)$ to $(-0.98, 1.99, 3.03)$

Solution:

$$dx = 0.02, \quad dy = -0.01, \quad dz = 0.03$$

$$df = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz$$

$$= (y+z)dx + (x+z)dy + (x+y)dz$$

At the given values,

$$df = (2+3)(0.02) + (-1+3)(-0.01) + (-1+2)(0.03) = 0.11$$