MATH 267 CHAPTER 16 PARTIAL DIFFERENTIATION

16.3 PARTIAL DERIVATIVES

Definition Let *f* be a function of two variables. The **<u>first partial derivatives</u> of *f* with respect to *x* and *y*** are the functions f_x and f_y such that

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Remark To find f_x, regard *y* as a constant and differentiate with respect to *x*.
To find f_y, regard *x* as a constant and differentiate with respect to *y*.

Notations If $w = f(x,y)$ then

$$f_x = \frac{\partial f}{\partial x}, \qquad f_y = \frac{\partial f}{\partial y}$$

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = \frac{\partial w}{\partial x} = w_x$$

$$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = \frac{\partial w}{\partial y} = w_y$$

Exercises Find the first partial derivatives of *f*.

2) $f(x,y) = \left(x^3 - y^2\right)^5$

Solution:

$$f_x(x,y) = 5\left(x^3 - y^2\right)^4 3x^2 = 15x^2\left(x^3 - y^2\right)^4 \qquad f_y(x,y) = 5\left(x^3 - y^2\right)^4(-2y) = -10y\left(x^3 - y^2\right)^4$$

4) $f(s,t) = \dfrac{t}{s} - \dfrac{s}{t}$

Solution:

$$\frac{\partial}{\partial s} f(s,t) = -\frac{t}{s^2} - \frac{1}{t} \qquad \frac{\partial}{\partial t} f(s,t) = \frac{1}{s} + \frac{s}{t^2}$$

6) $f(x,y) = e^x \ln xy$

Solution:

$$\frac{\partial f}{\partial x} = e^x \frac{\partial}{\partial x}(\ln xy) + \ln xy \frac{\partial}{\partial x}\left(e^x\right) = e^x \frac{1}{xy}\cdot y + \ln xy \cdot \left(e^x\right) = e^x\left(\frac{1}{x} + \ln xy\right)$$

$$\frac{\partial f}{\partial y} = e^x \frac{\partial}{\partial y}(\ln xy) = e^x \frac{1}{xy}\cdot x = \frac{e^x}{y}$$

14) $f(x,y,t) = \dfrac{x^2 - t^2}{1 + \sin 3y}$

Solution:

$$f_x = \frac{2x}{1+\sin 3y} \qquad f_y = -\frac{3\left(x^2 - t^2\right)\cos 3y}{(1+\sin 3y)^2} \qquad f_t = -\frac{2t}{1+\sin 3y}$$

Theorem Let S be the graph of $z = f(x,y)$, and let $P(a,b,f(a,b))$ be a point on S at which f_x and f_y exist. Let C_1 and C_2 be the traces of S on the planes $x = a$ and $y = b$, respectively, and let l_1 and l_2 be the tangent lines to C_1 and C_2 at P.

- **(i)** The slope of l_1 in the plane $x = a$ is $f_y(a,b)$.
- **(ii)** The slope of l_2 in the plane $y = b$ is $f_x(a,b)$.

Remark Similar definitions, etc., hold for first partial derivatives of functions of three or more variables.

Second Partial Derivatives

$$\frac{\partial}{\partial x} f_x = (f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
$$\frac{\partial}{\partial y} f_x = (f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$\frac{\partial}{\partial x} f_y = (f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$
$$\frac{\partial}{\partial y} f_y = (f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

f_{xy} and f_{yx} are called **mixed second partial derivatives** of f, or simply **mixed partials** of f.

Theorem Let f be a function of two variables x and y. If f, f_x, f_y, f_{xy}, and f_{yx} are continuous on an open region R, then $f_{xy} = f_{yx}$ throughout R.

Note: If $w = f(x,y,z)$ and f has continuous first and second partial derivatives, then the following hold:

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}, \qquad \frac{\partial^2 w}{\partial z \partial x} = \frac{\partial^2 w}{\partial x \partial z}, \qquad \frac{\partial^2 w}{\partial z \partial y} = \frac{\partial^2 w}{\partial y \partial z}$$

Exercises Verify that $w_{xy} = w_{yx}$.

20) $w = \dfrac{x^2}{x+y}$

Solution:

$$w_x = \frac{(x+y)2x - x^2 \cdot 1}{(x+y)^2} = \frac{x^2 + 2xy}{(x+y)^2}$$
$$w_{xy} = \frac{(x+y)^2 2x - (x^2 + 2xy)2(x+y) \cdot 1}{(x+y)^4}$$
$$= \frac{2x(x+y) - 2(x^2 + 2xy)}{(x+y)^3} = -\frac{2xy}{(x+y)^3}$$

$$w_y = -\frac{x^2}{(x+y)^2}$$
$$w_{yx} = -\frac{(x+y)^2 2x - x^2 2(x+y)}{(x+y)^4}$$
$$= -\frac{2x(x+y) - 2x^2}{(x+y)^3} = -\frac{2xy}{(x+y)^3}$$

A function f of x and y is **harmonic** if

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$$

throughout the domain of f. Prove that the given function is harmonic.

34) $f(x,y)=\arctan\frac{y}{x}$

Solution:

$$\frac{\partial f}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)=\frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)=\frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{2xy}{\left(x^2+y^2\right)}$$

$$\frac{\partial^2 f}{\partial y^2}=-\frac{2xy}{\left(x^2+y^2\right)}$$

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$$

40) The ideal gas law may be stated as $PV=knT$, where n is the number of moles of gas, V is the volume, T is the temperature, P is the pressure, and k is a constant. Show that

$$\frac{\partial V}{\partial T}\frac{\partial T}{\partial P}\frac{\partial P}{\partial V}=-1.$$

Solution:

$$PV=knT\Rightarrow V=\frac{knT}{P}\Rightarrow\frac{\partial V}{\partial T}=\frac{kn}{P}$$

$$PV=knT\Rightarrow T=\frac{PV}{kn}\Rightarrow\frac{\partial T}{\partial P}=\frac{V}{kn}$$

$$PV=knT\Rightarrow P=\frac{knT}{V}\Rightarrow\frac{\partial P}{\partial V}=-\frac{knT}{V^2}$$

$$\frac{\partial V}{\partial T}\frac{\partial T}{\partial P}\frac{\partial P}{\partial V}=\frac{kn}{P}\frac{V}{kn}\left(-\frac{knT}{V^2}\right)=-\frac{1}{P}\frac{PV^2}{V^2}=-1$$

50) The surface of a certain lake is represented by a region D in an xy-plane such that the depth under the point corresponding to (x,y) is given by $f(x,y)=300-2x^2-3y^2$, where x, y, and $f(x,y)$ are in feet. If a water skier is in the water at the point $(4,9)$, find the instantaneous rate at which the depth changes in the direction of

a) the x-axis

b) the y-axis

Solution:

$$f(x,y)=300-2x^2-3y^2$$

$$f_x=-4x\big|_4=-16\ (\text{depth in ft})/(\text{distance in ft})$$

$$f_y=-6y\big|_9=-54\ (\text{depth in ft})/(\text{distance in ft})$$

52) An object is situated in a rectangular coordinate system such that the temperature T at the point $P(x,y,z)$ is given by $T = 4x^2 - y^2 + 16z^2$, where T is in degrees and x, y, and z are in centimeters. Find the instantaneous rate of change of T with respect to distance at the point $P(4,-2,1)$ in the direction of

a) the x-axis
b) the y-axis
c) the z-axis

Solution:

$$T = 4x^2 - y^2 + 16z^2$$
$$T_x = 8x\big|_4 = 32 \text{ deg/cm}$$
$$T_y = -2y\big|_{-2} = 4 \text{ deg/cm}$$
$$T_z = 32z\big|_1 = 32 \text{ deg/cm}$$

54) The analysis of certain electrical circuits involves the formula

$$I = \frac{V}{\sqrt{R^2 + L^2\omega^2}},$$

where I is the current, V is the voltage, R the resistance, L the inductance, and ω a positive constant. Find and interpret $\frac{\partial I}{\partial R}$ and $\frac{\partial I}{\partial L}$.

Solution:

$$I = \frac{V}{\sqrt{R^2 + L^2\omega^2}}$$

$$\frac{\partial I}{\partial R} = -\frac{RV}{\left(R^2 + L^2\omega^2\right)^{3/2}}$$ the rate of change of current wrt resistance

$$\frac{\partial I}{\partial L} = -\frac{\omega^2 LV}{\left(R^2 + L^2\omega^2\right)^{3/2}}$$ the rate of change of current wrt inductance

64) Let C be the trace of the graph of $z = \sqrt{36 - 9x^2 - 4y^2}$ on the plane $y = 2$. Find parametric equations of the tangent line l to C at the point $P\left(1,2,\sqrt{11}\right)$. Sketch the surface, C, and l.

Solution: C is the intersection of the plane $y = 2$ with the upper half of the ellipsoid

$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{36} = 1$ and l is the tangent line to C at the point $P\left(1,2,\sqrt{11}\right)$.

$$z = \sqrt{36 - 9x^2 - 4y^2}$$
$z_x = -9x\left(36 - 9x^2 - 4y^2\right)^{-3/2} \Rightarrow$ slope of l at $P\left(1,2,\sqrt{11}\right)$ is $-9/\sqrt{11}$

Equation of l is $z - \sqrt{11} = -\frac{9}{\sqrt{11}}(x-1)$ or $z = -\frac{9}{\sqrt{11}}x + \frac{20}{\sqrt{11}}$

Thus, parametric equations of l are

$$x = t, \quad y = 2, \quad z = \frac{1}{\sqrt{11}}(-9t + 20).$$