MATH 267 CHAPTER 16 PARTIAL DIFFERENTIATION

16.2 LIMITS AND CONTINUITY

Limit Notation $\lim_{(x,y)\to(a,b)} f(x,y) = L$, or $f(x,y)\to L$ as $(x,y)\to(a,b)$

Definition Let a function *f* of two variables be defined throughout the interior of a circle with center (a,b), except possibly at (a,b) itself. The statement

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

means that for every $\in > 0$ there is a $\delta > 0$ such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \in .$$

Exercises

Find the limit.

2) $\lim_{(x,y)\to(2,1)} \dfrac{4+x}{2-y}$

Solution:

$$\lim_{(x,y)\to(2,1)} \frac{4+x}{2-y} = \frac{4+2}{2-1} = 6$$

4) $\lim_{(x,y)\to(-1,3)} \dfrac{y^2+x}{(x-1)(y+2)}$

Solution:

$$\lim_{(x,y)\to(-1,3)} \frac{y^2+x}{(x-1)(y+2)} = \frac{3^2+(-1)}{(-1-1)(3+2)} = \frac{8}{-10} = -\frac{4}{5}$$

6) $\lim_{(x,y)\to(1,2)} \dfrac{xy-y}{x^2-x+2xy-2y}$

Solution:

$$\lim_{(x,y)\to(1,2)} \frac{xy-y}{x^2-x+2xy-2y} = \frac{(1)(2)-2}{1^2-1+2(1)(2)-2(2)} = \frac{0}{0} \quad \text{I.F.}$$

Factor:

$$\lim_{(x,y)\to(1,2)} \frac{y(x-1)}{(x+2y)(x-1)} = \lim_{(x,y)\to(1,2)} \frac{y}{x+2y} = \frac{2}{1+2(2)} = \frac{2}{5}$$

8) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 - x^2y + xy^2 - y^3}{x^2 + y^2}$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - x^2y + xy^2 - y^3}{x^2 + y^2} = \frac{0}{0} \quad \text{I.F.}$$

Factor:

$$\lim_{(x,y)\to(0,0)} \frac{\left(x^2 + y^2\right)(x - y)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} (x - y) = 0$$

<u>Two-path rule</u>: If two different paths to a point $P(a,b)$ produce two different limiting values for *f*, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Show that the limit does not exist.

12) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - 2xy + 5y^2}{3x^2 + 4y^2}$

Solution:

Let $(x,y)\to(0,0)$ along the *x*-axis. Then the *y* coordinate is always zero and we get

$$\frac{x^2 - 2xy + 5y^2}{3x^2 + 4y^2} = \frac{x^2}{3x^2} = \frac{1}{3}$$

Let $(x,y)\to(0,0)$ along the *y*-axis. Then the *x* coordinate is always zero and we get

$$\frac{x^2 - 2xy + 5y^2}{3x^2 + 4y^2} = \frac{5y^2}{4y^2} = \frac{5}{4}.$$

By the two-path rule, the limit does not exist.

14) $\lim\limits_{(x,y)\to(2,1)} \dfrac{x^2 - 4x + 4}{xy - 2y - x + 2}$

Solution:

Let $(x,y)\to(2,1)$ along the line $y - 1 = m(x-2)$. Then

$$\begin{aligned}\lim_{(x,y)\to(2,1)} \frac{x^2 - 4x + 4}{xy - 2y - x + 2} &= \lim_{(x,y)\to(2,1)} \frac{x^2 - 4x + 4}{xy - 2y - x + 2} \\ &= \lim_{(x,y)\to(2,1)} \frac{(x-2)^2}{(y-1)(x-2)} \\ &= \lim_{(x,y)\to(2,1)} \frac{(x-2)^2}{m(x-2)^2} \\ &= \frac{1}{m}\end{aligned}$$

Since we get different values of *m* will give different values for the limit, the limit DNE.

16) $\lim_{(x,y)\to(0,0)} \frac{3xy}{5x^4+2y^4}$

Solution:

Let $(x,y)\to(0,0)$ along the line $y = mx$. Then

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{3xy}{5x^4+2y^4} &= \lim_{(x,y)\to(0,0)} \frac{3x(mx)}{5x^4+2(mx)^4} \\
&= \lim_{(x,y)\to(0,0)} \frac{3mx^2}{5x^4+2m^4x^4} \\
&= \lim_{(x,y)\to(0,0)} \frac{3mx^2}{x^4(5+2m^4)} \\
&= \lim_{(x,y)\to(0,0)} \frac{3m}{x^2(5+2m^4)} = \infty
\end{aligned}$$

Use polar coordinates to find the limit, if it exists.

22) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

Solution:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} &= \lim_{r\to 0} \frac{(r\cos\theta)^3-(r\sin\theta)^3}{r^2} \\
&= \lim_{r\to 0} \frac{r^3(\cos^3\theta-\sin^3\theta)}{r^2} \\
&= \lim_{r\to 0} r(\cos^3\theta-\sin^3\theta) = 0
\end{aligned}$$

24) $\lim_{(x,y)\to(0,0)} \frac{\sinh(x^2+y^2)}{x^2+y^2}$

Solution:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{\sinh(x^2+y^2)}{x^2+y^2} &= \lim_{r\to 0} \frac{\sinh(r^2)}{r^2} = \frac{0}{0} \quad \text{I.F.} \\
&= \lim_{r\to 0} \frac{2r\cosh(r^2)}{2r} \\
&= \lim_{r\to 0} \cosh(r^2) = 1
\end{aligned}$$

Regions of the *xy* – plane

open disk – consists of all points that lie *inside* a circle

closed disk – contains both the points inside and *on* the circle

A point (a,b) is an **interior point** of a region R if there is an open disk with center (a,b) that lies completely within R. A point (a,b) is a **boundary point** of R if every disk with center (a,b) contains points that are in R and points that are not in R.

Similar concepts can be defined for a 3-dimensional region R using spheres instead of disks.

A region is **closed** if it contains all its boundary points. A region is **open** if it contains none of its boundary points, i.e. every point of the region is an interior point. A region that contains some, but not all, of its boundary points is neither open nor closed.

Definition A function f of two variables is **continuous** at an interior point (a,b) of its domain if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)\,.$$

Definition A function f is **continuous on its domain** D if it is continuous at every pair (a,b) in D.

Definition Let a function f of three variables be defined throughout the interior of a sphere with center (a,b,c), except possibly at (a,b,c) itself. The statement

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$$

means that for every $\in>0$ there is a $\delta>0$ such that

$$0<\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}<\delta \Rightarrow \left| f(x,y,z)-L\right| <\in\,.$$

Definition A function f of three variables is **continuous** at an interior point of a region if

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)\,.$$

Theorem If a function f of two variables is continuous at (a,b) and a function g of one variable is continuous at $f(a,b)$, then the function h defined by $h(x,y) = g(f(x,y))$ is continuous at (a,b).

18) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2+3y^2+z^2}{x^2+y^2+z^2}$

Solution:

Let $(x,y,z)\to(0,0,0)$ along the line l that is parallel to the vector $\langle a,b,c\rangle$. Parametric equations for l are

$$x = at,\quad y = bt,\quad z = ct,\quad t\in\mathbb{R}\,.$$

Thus,

$$\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2+3y^2+z^2}{x^2+y^2+z^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{2(at)^2+3(bt)^2+(ct)^2}{(at)^2+(bt)^2+(ct)^2}$$
$$= \lim_{(x,y,z)\to(0,0,0)} \frac{2a^2+3b^2+c^2}{a^2+b^2+c^2}$$

and we get different limits when we assign different values for a, b, and c.

OR

Let $(x,y,z)\to(0,0,0)$ along the y-axis and along the path $x = y = z$.

20) $\lim_{(x,y,z)\to(2,1,0)} \frac{(x+y+z-3)^5}{z^3(x-2)(y-1)}$

Solution:

Let $(x,y,z)\to(2,1,0)$ along the line *l* that is parallel to the vector $\langle a,b,c\rangle$. Parametric equations for *l* are

$$x = 2+at, \quad y = 1+bt, \quad z = ct, \quad t\in\mathbb{R}.$$

Thus,

$$\lim_{(x,y,z)\to(2,1,0)} \frac{(x+y+z-3)^5}{z^3(x-2)(y-1)} = \lim_{(x,y,z)\to(2,1,0)} \frac{(2+at+1+bt+ct-3)^5}{(ct)^3(2+at-2)(1+bt-1)}$$
$$= \lim_{(x,y,z)\to(2,1,0)} \frac{(at+bt+ct)^5}{c^3t^3(at)(bt)}$$
$$= \lim_{(x,y,z)\to(2,1,0)} \frac{(a+b+c)^5}{abc^3}$$

and we get different limits when we assign different values for *a*, *b*, and *c*.

Describe the set of all points in the *xy*-plane at which *f* is continuous.

26) $f(x,y) = \frac{xy}{x^2-y^2}$

Solution:

$f(x,y) = \frac{xy}{x^2-y^2}$ is continuous at all points on the plane for which $x^2 \neq y^2$.

28) $f(x,y) = \sqrt{25-x^2-y^2}$

Solution:

We want $25-x^2-y^2 \geq 0$ or $x^2+y^2 \leq 25$ or all points inside or on the circle $x^2+y^2 = 25$.

Describe the set of all points in an *xyz*-coordinate system at which *f* is continuous.

30) $f(x,y,z) = \sqrt{xy}\tan z$

Solution:

We want $xy \geq 0$ and $z \neq n\pi/2,$ n odd integer.

32) $f(x,y,z) = \sqrt{4-x^2-y^2-z^2}$

Solution:

We want: $4-x^2-y^2-z^2 \geq 0 \Leftrightarrow x^2+y^2+z^2 \leq 4$

This is the set of points that are inside or on the sphere $x^2+y^2+z^2 = 4$.

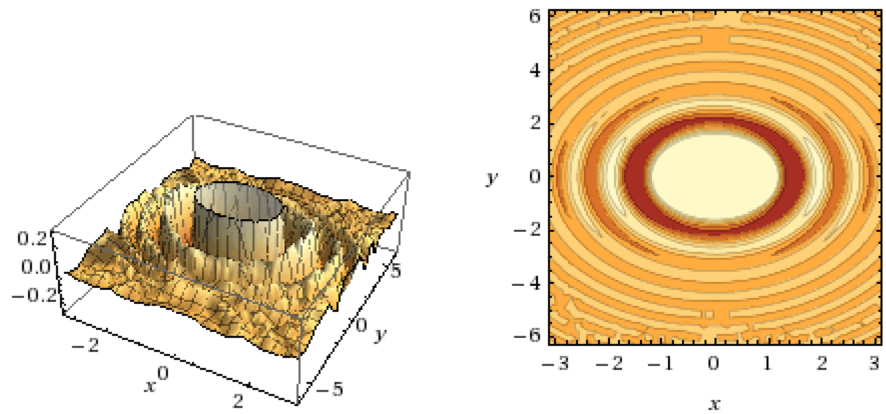
34) Use polar coordinates to investigate $\lim_{(x,y)\to(0,0)} \frac{\sin(2x^2+y^2)}{x^2+y^2}$.

Solution:

$$\lim_{r\to 0}\frac{\sin(2x^2+y^2)}{x^2+y^2} = \lim_{r\to 0}\frac{\sin(r^2\cos^2\theta + r^2)}{r^2} = \frac{0}{0}$$

$$= \lim_{r\to 0}\frac{(2r\cos^2\theta + 2r)\cos(r^2\cos^2\theta + r^2)}{2r}$$

$$= \cos^2\theta + 1$$



Find $h(x,y) = g(f(x,y))$ and use Theorem 16.7 to determine where h is continuous.

36) $f(x,y) = 3x + 2y - 4;\ \ g(t) = \ln(t+5)$

Solution:

$f(x,y) = 3x + 2y - 4;\ \ g(t) = \ln(t+5)$

$h(x,y) = g(f(x,y)) = \ln(3x + 2y - 4 + 5) = \ln(3x + 2y + 1)$

h is continuous on $\{(x,y) | 3x + 2y > -1\}$

38) $f(x,y) = y\ln x;\ \ \ g(w) = e^w$

Solution:

$f(x,y) = y\ln x;\ \ \ g(w) = e^w$

$h(x,y) = g(f(x,y)) = e^{y\ln x} = x^y$

h continuous on $\{(x,y) | x > 0\}$

40) If $f(x,y,z) = 2x + ye^z$ and $g(t) = t^2$, find $g(f(x,y,z))$.

Solution:

$f(x,y,z) = 2x + ye^z$ and $g(t) = t^2$

$g(f(x,y,z)) = (2x + ye^z)^2$

42) If $f(x,y) = 2x + y$, find $f(f(x,y), f(x,y))$.

Solution:

$f(x,y) = 2x + y$

$f(f(x,y), f(x,y)) = 2(2x + y) + 2x + y = 6x + 3y$