MATH 267 CHAPTER 16 PARTIAL DIFFERENTIATION

16.1 FUNCTIONS OF SEVERAL VARIABLES

Definition Let *D* be a set of ordered pairs of real numbers. A **function *f* of two variables** is a correspondence that assigns to each pair (x, y) in *D* exactly one real number, denoted by $f(x, y)$. The set *D* is the **domain** of *f*. The **range** of *f* consists of all real numbers $f(x, y)$, where (x, y) is in *D*.

Exercises

Describe the domain of *f*, and find the indicated function values.

2) $f(x,y) = \dfrac{y+2}{x}, \quad f(3,1), \quad f(1,3), \quad f(2,0)$

Solution:

$$f(x,y) = \frac{y+2}{x}, \quad f(3,1), \quad f(1,3), \quad f(2,0)$$
$$D_f = \{(x,y) | x \neq 0\}$$
$$f(3,1) = \frac{1+2}{3} = 1$$
$$f(1,3) = \frac{3+2}{1} = 5$$
$$f(2,0) = \frac{0+2}{2} = 1$$

4) $f(r,s) = \sqrt{1-r} - e^{r/s}, \quad f(1,1), \quad f(0,4), \quad f(-3,3)$

Solution:

$$f(r,s) = \sqrt{1-r} - e^{r/s}, \quad f(1,1), \quad f(0,4), \quad f(-3,3)$$
$$D_f = \{(r,s) | r \leq 1 \text{ and } s \neq 0\}$$
$$f(1,1) = \sqrt{1-1} - e^{1/1} = -e$$
$$f(0,4) = \sqrt{1-0} - e^{0/4} = 1-1 = 0$$
$$f(-3,3) = \sqrt{1-(-3)} - e^{-3/3} = 2 - e^{-1} = 2 - 1/e$$

6) $f(x,y,z) = 2 + \tan x + y \sin z; \quad f(\pi/4, 4, \pi/6), \quad f(0,0,0)$

Solution:

$$f(x,y,z) = 2 + \tan x + y \sin z; \quad f(\pi/4, 4, \pi/6), \quad f(0,0,0)$$
$$D_f = \{(x,y,z) | x \neq n\pi/2,\ n \text{ odd}\}$$
$$f(\pi/4, 4, \pi/6) = 2 + \tan(\pi/4) + 4\sin(\pi/6) = 2 + 1 + 4(1/2) = 5$$
$$f(0,0,0) = 2 + \tan 0 + 4 \sin 0 = 2$$

Remarks

(i) Formulas may be used to define functions of two variables. For example, $V = \pi r^2 h$. Here, r and h are the **independent variables** and V is the **dependent variable**.

(ii) A function f of three variables is defined similarly, except that the domain D is a subset of $\mathbb{R}^3$.

(iii) The **graph** of f is the graph of the equation $z = f(x, y)$ in an $xyz-$coordinate system, and is usually a surface S of some type.

(iv) If the trace of the graph of $z = f(x, y)$ on the plane $z = k$ is projected onto the $xy-$plane, a curve C with equation $f(x, y) = k$ is obtained. Note that if a point $(x, y, 0)$ moves along C, the corresponding $f(x, y)$ always equal k. C is called a **level curve** of f.

(v) If f is a function of three variables x, y, and z, then the **level surfaces** of f are the graphs of $f(x, y, z) = k$ for suitable values of k.

Exercises

Sketch the graph of f.

8) $f(x,y) = 4 - x^2 - 4y^2$

Solution:

$$f(x,y) = 4 - x^2 - 4y^2$$

$$z = 4 - x^2 - 4y^2$$

$xy-$trace: $0 = 4 - x^2 - 4y^2$

$$x^2 + 4y^2 = 4$$

$$\frac{x^2}{4} + y^2 = 1 \quad \text{ellipse}$$

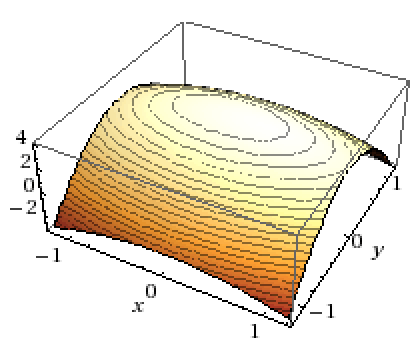
$xz-$trace: $z = 4 - x^2$ parabola

$yz-$trace: $z = 4 - 4y^2$ parabola

traces on planes parallel to the $z-$axis:

$z = k$: $k = 4 - x^2 - 4y^2$

$x^2 + 4y^2 = 4 - k$ ellipses for $k < 4$, a point for $k = 4$



10) $f(x,y)=\frac{1}{6}\sqrt{9x^2+4y^2}$

Solution:

$f(x,y)=\frac{1}{6}\sqrt{9x^2+4y^2}$

$z=\frac{1}{6}\sqrt{9x^2+4y^2}$

$6z=\sqrt{9x^2+4y^2}$

$36z^2=9x^2+4y^2$

$9x^2+4y^2-36z^2=0$

$xy-$trace: $9x^2+4y^2=0$ a point

$xz-$trace: $9x^2-36z^2=0$

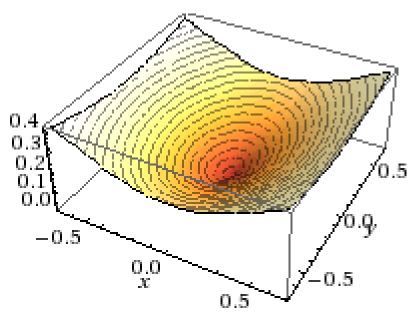
$z=\pm\frac{1}{2}x$ 2 lines

$yz-$trace: $4y^2-36z^2=0$

$z=\pm\frac{1}{3}y$ 2 lines

traces on planes parallel to the $z-$axis:

$z=k>0$: $36k^2=9x^2+4y^2$ ellipses



12) $f(x,y)=\sqrt{72+4x^2-9y^2}$

Solution:

$f(x,y)=\sqrt{72+4x^2-9y^2}$

$z=\sqrt{72+4x^2-9y^2}$

$z^2=72+4x^2-9y^2$

$-4x^2+9y^2+z^2=72$

$xy-$trace: $-4x^2+9y^2=72$

$-\dfrac{x^2}{18}+\dfrac{y^2}{8}=1$ hyperbola with vertices on the $y-$axis

$xz-$trace: $-4x^2+z^2=72$

$-\dfrac{x^2}{18}+\dfrac{z^2}{72}=1$ hyperbola with vertices on the $z-$axis

$yz-$trace: $9y^2+z^2=72$

$\dfrac{y^2}{8}+\dfrac{z^2}{72}=1$ ellipse with major axis along the $z-$axis

traces on planes parallel to the $z-$axis:

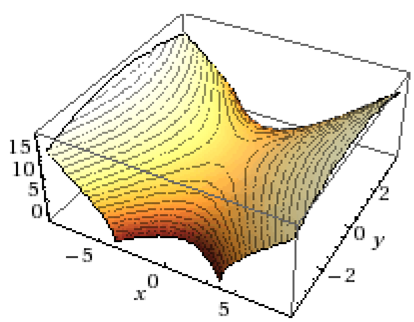
$z=k\geq 0$: $k^2=72+4x^2-9y^2$

$4x^2-9y^2=k^2-72$

hyperbolas with transverse axis on the $x-$axis for $k>6\sqrt{2}$

hyperbolas with transverse axis on the $y-$axis for $0<k<6\sqrt{2}$

two lines for $k=6\sqrt{2}$



14) $f(x,y)=\sqrt{x^2+4y^2+25}$

Solution:

$f(x,y)=\sqrt{x^2+4y^2+25}$

$z=\sqrt{x^2+4y^2+25}$

$z^2=x^2+4y^2+25$

$z^2-x^2-4y^2=25$

$xy-$trace: $x^2+4y^2=-25$ none

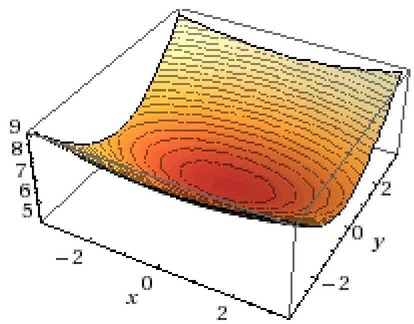
$xz-$trace: $z^2-x^2=25$ hyperbola with vertices on the $z-$axis

$yz-$trace: $z^2-4y^2=25$ hyperbola with vertices on the $z-$axis

traces on planes parallel to the $z-$axis:

$z=k\ge 0$: $k^2=x^2+4y^2+25$

$x^2+4y^2=k^2-25$ ellipses for $k>5$; a point for $k=5$



Sketch the level curves of *f* for the given values of *k*.

16) $f(x,y)=3x-2y;\quad k=-4,0,6$

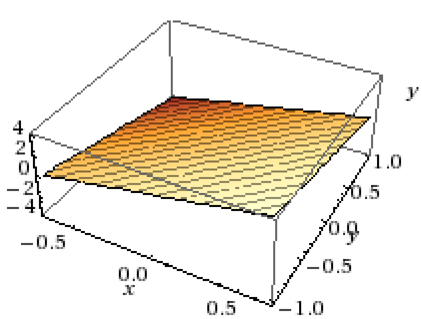
Solution:

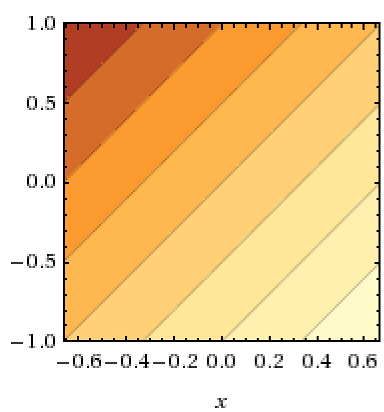
$f(x,y)=3x-2y;\quad k=-4,0,6$

$3x-2y=-4\Leftrightarrow \dfrac{x}{-4/3}+\dfrac{y}{2}=1$ line with slope $3/2$, through $(-4/3,0)$ and $(0,2)$

$3x-2y=0$ line with slope $3/2$, through $(0,0)$

$3x-2y=6\Leftrightarrow \dfrac{x}{2}-\dfrac{y}{3}=1$ line with slope $3/2$, through $(2,0)$ and $(0,3)$





18) $f(x,y) = xy; \qquad k = -4, 1, 4$

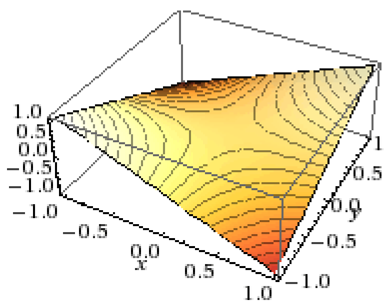
Solution:

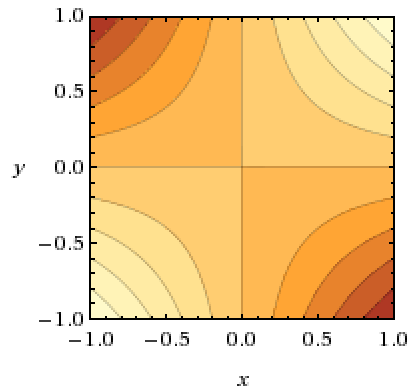
$f(x,y) = xy; \qquad k = -4, 1, 4$

$xy = -4 \Leftrightarrow y = -4/x$

$xy = 1 \Leftrightarrow y = 1/x$

$xy = 4 \Leftrightarrow y = 4/x$





20) $f(x,y) = 4x^2 + y^2; \qquad k = 4, 9, 16$

Solution:

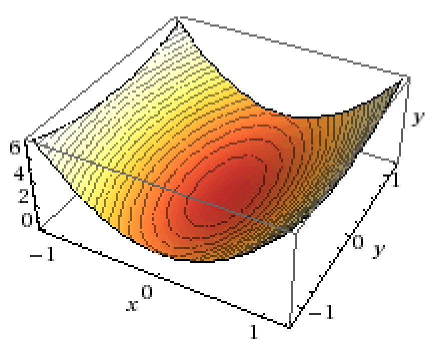
$f(x,y) = 4x^2 + y^2; \qquad k = 4, 9, 16$

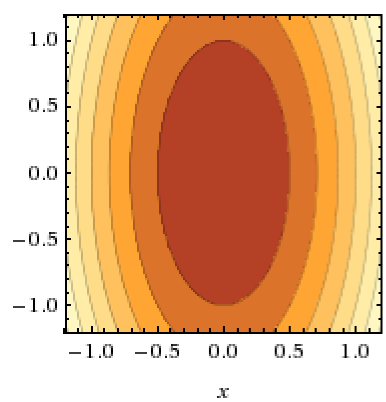
$4x^2 + y^2 = 4$

$4x^2 + y^2 = 9$

$4x^2 + y^2 = 16$

ellipses





Find an equation of the level curve of *f* that contains the point *P*.

22) $f(x,y) = (2x + y^2)e^{xy}$; $P(0,2)$

Solution:

$$f(x,y) = (2x + y^2)e^{xy}; \quad P(0,2)$$

$$f(0,2) = (2 \cdot 0 + 2^2)e^{0 \cdot 2} = 4$$

Thus, the equation of the level curve that contains $P(0,2)$ is $(2x + y^2)e^{xy} = 4$.

Find an equation of the level surface of *f* that contains the point *P*.

24) $f(x,y,z) = z^2y + x$; $P(1,4,-2)$

Solution:

$$f(x,y,z) = z^2y + x; \quad P(1,4,-2)$$

$$f(1,4,-2) = (-2)^2 4 + 1 = 17$$

Thus, the equation of the level surface is $z^2y + x = 17$.

Describe the level surfaces of *f* for the given values of *k*.

38) $f(x,y,z) = z + x^2 + 4y^2$; $k = -1, 0, 2$

Solution:

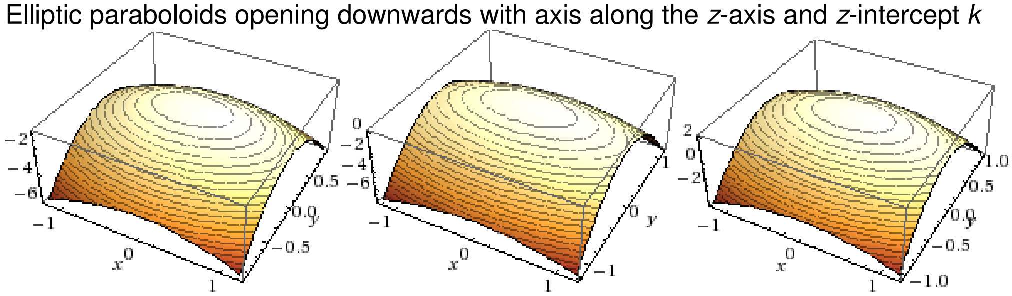
$$f(x,y,z) = z + x^2 + 4y^2; \quad k = -1, 0, 2$$

$$z + x^2 + 4y^2 = -1$$

$$z + x^2 + 4y^2 = 0$$

$$z + x^2 + 4y^2 = 2$$

Elliptic paraboloids opening downwards with axis along the *z*-axis and *z*-intercept *k*



44) If the voltage V at the point $P(x, y, z)$ is given by $V = \frac{6}{\left(x^2 + 4y^2 + 9z^2\right)^{1/2}}$,

a) describe the equipotential surfaces
b) find an equation of the equipotential surface $V = 120$

Solution:

a) Let $V = k$:

$$k = \frac{6}{\left(x^2 + 4y^2 + 9z^2\right)^{1/2}}$$

$$k\left(x^2 + 4y^2 + 9z^2\right)^{1/2} = 6$$

$$k^2\left(x^2 + 4y^2 + 9z^2\right) = 36$$

$$x^2 + 4y^2 + 9z^2 = \frac{36}{k^2}$$

The equipotential surfaces are ellipsoids centered at the origin.

b) Let $k = 120$:

$$x^2 + 4y^2 + 9z^2 = \frac{36}{120^2}$$

$$x^2 + 4y^2 + 9z^2 = \frac{1}{400}$$

$$400x^2 + 1600y^2 + 3600z^2 = 1$$

46) According to the *ideal gas law*, the pressure *P*, the volume *V*, and the temperature *T* of a confined gas are related by the formula $PV = kT$ for a constant *k*. Express *P* as a function of *V* and *T*, and describe the level curves associated with this function. What is the physical significance of these level curves?

Solution:

$$PV = kT$$

$$P = \frac{kT}{V}$$

Level curves: $\frac{kT}{V} = c$, c is a constant

$V = \frac{kT}{c}$, lines in the VT-plane passing through the origin

As a point moves on a level curve, the pressure is constant.

48) If x is wind velocity (in m/sec) and y is temperature (in ^{0}C), then the *windchill factor F* (in kcal/m^2) is given by

$$F = (33 - y)\left(10\sqrt{x} - x + 10.5\right).$$

a) Find the velocities and temperatures for which the windchill factor is 0. (Assume that $0 \le x \le 50$ and $-50 \le y \le 50$.)

b) If $F \ge 1400$, frostbite will occur on exposed human skin. Sketch the graph of the level curve $F = 1400$.

Solution:

a)

$$(33 - y)\left(10\sqrt{x} - x + 10.5\right) = 0$$

$$\Leftrightarrow 33 - y = 0 \quad \text{or} \quad 10\sqrt{x} - x + 10.5 = 0$$

$$\Leftrightarrow y = 33 \qquad \text{or} \qquad 10\sqrt{x} = x - 10.5$$

$$x \approx 120.1 \notin [0,50]$$

$F = 0$ when $y = 33$.

b) $1400 = (33 - y)\left(10\sqrt{x} - x + 10.5\right)$

52) The atmospheric pressure near ground level in a certain region is given by

$$p(x, y) = ax^2 + by^2 + c,$$

where *a, b,* and *c* are positive constants.

a) Describe the isobars in this region for pressures greater than *c*.

b) Is this a region of high or low pressure?

Solution:

a) Isobars: For $k > c$

$$ax^2 + by^2 + c = k$$

$$ax^2 + by^2 = k - c$$

$$\frac{x^2}{(k-c)/a} + \frac{y^2}{(k-c)/b} = 1 \quad \text{ellipses}$$

b) The atmospheric pressure increases as we move away from the origin. p is minimum at $(0,0)$ and increases with each level curve. Thus, there is an area of low pressure near the origin.