MATH 267 CHAPTER 15 VECTOR-VALUED FUNCTIONS AND SPACE CURVES

15.5 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

Theorem Let $\vec{r}(t)$ be the position vector of a moving point *P* at time *t*, and let *s* be an arc length parameter for the curve *C* determined by $\vec{r}(t)$. If $\vec{T}(s)$ and $\vec{N}(s)$ are the unit tangent and principal unit normal vectors and if *K* is the curvature of *C* at the point corresponding to *s*, then the velocity and acceleration of *P* may be written as

$$\vec{v}(t) = \frac{ds}{dt}\vec{T}(s)$$
$$\vec{a}(t) = \frac{d^2s}{dt^2}\vec{T}(s) + K\left(\frac{ds}{dt}\right)^2\vec{N}(s)$$

Proof:

$$\vec{T}(s) = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|} \Rightarrow \vec{r}\,'(t) = \left\|\vec{r}\,'(t)\right\|\vec{T}(s)$$

Since $\vec{r}\,'(t) = \vec{v}(t)$ and $\left\|\vec{r}\,'(t)\right\| = \left\|\vec{v}(t)\right\| = ds/dt$, we get the velocity formula

$$\vec{v}(t) = \frac{ds}{dt}\vec{T}(s).$$

Differentiating wrt *t* and using the chain rules in Ex 45 and 46 in 15.2, we get

$$\vec{a}(t) = \vec{v}\,'(t) = \frac{d^2s}{dt^2}\vec{T}(s) + \frac{ds}{dt}\frac{d}{dt}\vec{T}(s)$$
$$= \frac{d^2s}{dt^2}\vec{T}(s) + \frac{ds}{dt}\frac{ds}{dt}\vec{T}\,'(s)$$

From the remark following Def 15.20, $\vec{T}\,'(s) = K\vec{N}(s)$. Thus,

$$\vec{a}(t) = \frac{d^2s}{dt^2}\vec{T}(s) + K\left(\frac{ds}{dt}\right)^2\vec{N}(s) \quad .$$

Note: Let $v = \left\|\vec{v}(t)\right\| = ds/dt$. Then the formulas in the theorem can be written as

$$\vec{v}(t) = v\vec{T}(s)$$
$$\vec{a}(t) = \underbrace{\frac{dv}{dt}}_{\text{tangential component}}\vec{T}(s) + \underbrace{Kv^2}_{\text{normal component}}\vec{N}(s)$$
$$\vec{a}(t) = a_T\vec{T}(s) + a_N\vec{N}(s)$$

Next, we want to derive formulas for $a_{\vec{T}}$ and $a_{\vec{N}}$ that depend only on $\vec{r}(t)$.

$$\vec{v}(t)\cdot\vec{a}(t)=\left[\frac{ds}{dt}\vec{T}(s)\right]\cdot\left[\frac{d^2s}{dt^2}\vec{T}(s)+K\left(\frac{ds}{dt}\right)^2\vec{N}(s)\right]$$

$$=\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\underbrace{\left[\vec{T}(s)\cdot\vec{T}(s)\right]}_{\substack{1\text{ since}\\ \|\vec{T}(s)\|=1}}+K\left(\frac{ds}{dt}\right)^3\left[\underbrace{\vec{T}(s)\cdot\vec{N}(s)}_{0}\right]$$

$$\vec{v}(t)\cdot\vec{a}(t)=\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)$$

$$\vec{r}\,'(t)\cdot\vec{r}\,''(t)=\left\|\vec{r}\,'(t)\right\|\frac{d^2s}{dt^2}=\left\|\vec{r}\,'(t)\right\|a_{\vec{T}}$$

Thus,

$$a_{\vec{T}}=\frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\left\|\vec{r}\,'(t)\right\|}$$

Tangential component of acceleration $\quad a_{\vec{T}}=\dfrac{d^2s}{dt^2}=\dfrac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\left\|\vec{r}\,'(t)\right\|}$

Now if we take the cross product of $\vec{a}(t)$ and $\vec{v}(t)$:

$$\vec{v}(t)\times\vec{a}(t)=\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\underbrace{\left[\vec{T}(s)\times\vec{T}(s)\right]}_{0}+K\left(\frac{ds}{dt}\right)^3\left[\underbrace{\vec{T}(s)\times\vec{N}(s)}_{\substack{1\text{ since}\\ \|\vec{T}(s)\times\vec{N}(s)\|=1}}\right]$$

$$\left\|\vec{v}(t)\times\vec{a}(t)\right\|=K\left(\frac{ds}{dt}\right)^3=K\left(\frac{ds}{dt}\right)^2\left(\frac{ds}{dt}\right)=a_{\vec{N}}\left(\frac{ds}{dt}\right)$$

Thus,

$$a_{\vec{N}}=\frac{\left\|\vec{v}(t)\times\vec{a}(t)\right\|}{ds/dt}=\frac{\left\|\vec{r}\,'(t)\times\vec{r}\,''(t)\right\|}{\left\|\vec{r}\,'(t)\right\|}$$

Normal component of acceleration $\quad a_{\vec{N}}=K\left(\dfrac{ds}{dt}\right)^2=\dfrac{\left\|\vec{r}\,'(t)\times\vec{r}\,''(t)\right\|}{\left\|\vec{r}\,'(t)\right\|}$

Theorem Alternative formula for $a_{\vec{N}}$

$$a_{\vec{N}}=\sqrt{\left\|\vec{a}\right\|^2-a_{\vec{T}}^2}$$

Proof:

$$\vec{a}=a_{\vec{T}}\vec{T}+a_{\vec{N}}\vec{N}$$

$$\left\|\vec{a}\right\|^2=\vec{a}\cdot\vec{a}=\left(a_{\vec{T}}\vec{T}+a_{\vec{N}}\vec{N}\right)\cdot\left(a_{\vec{T}}\vec{T}+a_{\vec{N}}\vec{N}\right)$$

$$=a_{\vec{T}}^2\underbrace{\left(\vec{T}\cdot\vec{T}\right)}_{1}+2a_{\vec{N}}a_{\vec{T}}\underbrace{\left(\vec{N}\cdot\vec{T}\right)}_{0}+a_{\vec{N}}^2\underbrace{\left(\vec{N}\cdot\vec{N}\right)}_{1}$$

$$=a_{\vec{T}}^2+a_{\vec{N}}^2$$

Solve for $a_{\vec{N}}$.

Theorem Let a space curve C have the parametrization $x = f(t), \quad y = g(t), \quad z = h(t)$, where f'', g'', and h'' exist. The curvature K at the point $P(x,y,z)$ on C is

$$K = \frac{\left\| \vec{r}'(t) \times \vec{r}''(t) \right\|}{\left\| \vec{r}'(t) \right\|^3} = \frac{a_{\vec{N}}}{\left\| r'(t) \right\|^2}$$

Proof:

From $a_{\vec{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{\left\| \vec{r}'(t) \times \vec{r}''(t) \right\|}{\left\| \vec{r}'(t) \right\|}$ we can solve for K and get the result using the fact that $ds/dt = \left\| \vec{r}'(t) \right\|$.

Exercises

Find general formulas for the tangential and normal components of acceleration and for the curvature of the curve C determined by $\vec{r}(t)$.

2) $\vec{r}(t) = \left(2t^2 - 1\right)\hat{i} + 5t\hat{j}$

Solution:

$$\vec{r}(t) = \left(2t^2 - 1\right)\hat{i} + 5t\hat{j}$$

$$\vec{r}'(t) = 4t\hat{i} + 5\hat{j}; \qquad \left\| \vec{r}'(t) \right\| = \sqrt{16t^2 + 25}$$

$$\vec{r}''(t) = 4\hat{i}$$

$$a_{\vec{T}} = \frac{d^2s}{dt^2} = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\left\| \vec{r}'(t) \right\|} = \frac{16t}{\sqrt{16t^2 + 25}}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 4 & 5 & 0 \\ 4 & 0 & 0 \end{vmatrix} = -20k$$

$$a_{\vec{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{\left\| \vec{r}'(t) \times \vec{r}''(t) \right\|}{\left\| \vec{r}'(t) \right\|} = \frac{20}{\sqrt{16t^2 + 25}}$$

$$K = \frac{a_{\vec{N}}}{\left\| \vec{r}'(t) \right\|^2} = \frac{20}{\left(16t^2 + 25\right)^{3/2}}$$

8) $\vec{r}(t) = e^t\left(\sin t\ \hat{i} + \cos t\,\hat{j} + k\right)$

Solution:

$$\vec{r}(t) = e^t\left(\sin t\ \hat{i} + \cos t\,\hat{j} + k\right)$$

$$\vec{r}'(t) = \left(e^t \cos t + e^t \sin t\right)\hat{i} + \left(e^t \cos t - e^t \sin t\right)\hat{j} + e^t k$$

$$\vec{r}''(t) = \left(2e^t \cos t\right)\hat{i} - \left(2e^t \sin t\right)\hat{j} + e^t k$$

$$\left\| \vec{r}'(t) \right\| = \sqrt{\left(e^t \cos t + e^t \sin t\right)^2 + \left(e^t \cos t - e^t \sin t\right)^2 + e^{2t}}$$

$$= e^t\sqrt{\left(1 + 2\cos t \sin t\right) + \left(1 - 2\cos t \sin t\right) + 1}$$

$$= \sqrt{3}e^t$$

$$a_{\vec{T}} = \frac{d^2s}{dt^2} = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\left\| \vec{r}'(t) \right\|} = \frac{2e^{2t}\cos^2 t + \cancel{2e^{2t}\cos t}\sin t - \cancel{2e^{2t}\cos t}\sin t + 2e^{2t}\sin^2 t + e^{2t}}{\sqrt{3}e^t}$$

$$= \frac{3e^{2t}}{\sqrt{3}e^t} = \sqrt{3}e^t$$

$$\vec{r}'(t)\times\vec{r}''(t)=\begin{vmatrix}\hat{i} & \hat{j} & k\\ e^t\cos t+e^t\sin t & e^t\cos t-e^t\sin t & e^t\\ 2e^t\cos t & -2e^t\sin t & e^t\end{vmatrix}$$

$$=\left(e^{2t}\left(\cos t-\sin t\right)+2e^{2t}\sin t\right)\hat{i}-\left(e^{2t}\left(\cos t+\sin t\right)-2e^{2t}\cos t\right)\hat{j}+$$

$$2e^{2t}\left(-\sin t\cos t-\sin^2 t-\cos^2 t+\cos t\sin t\right)k$$

$$=e^{2t}\left(\cos t+\sin t\right)\hat{i}+e^{2t}\left(\cos t-\sin t\right)\hat{j}-2e^{2t}\widehat{k}$$

$$a_{\vec{N}}=K\left(\frac{ds}{dt}\right)^2=\frac{\left\|\vec{r}'(t)\times\vec{r}''(t)\right\|}{\left\|\vec{r}'(t)\right\|}=\frac{\left\|e^{2t}\left(\cos t+\sin t\right)\hat{i}+e^{2t}\left(\cos t-\sin t\right)\hat{j}-2e^{2t}\hat{k}\right\|}{\sqrt{3}e^t}$$

$$=\frac{e^{2t}\sqrt{6}}{\sqrt{3}e^t}=\sqrt{2}e^t$$

$$K=\frac{a_{\vec{N}}}{\left\|\vec{r}'(t)\right\|^2}=\frac{\sqrt{2}e^t}{3e^{2t}}=\frac{\sqrt{2}}{3e^t}$$

10) A point moves along the graph of $y=2x^3-x$ such that the horizontal component of velocity is always 3. Find the tangential and normal components of acceleration at $P(1,1)$.

Solution:

$y=2x^3-x$

$\vec{r}(t)=f(t)\hat{i}+g(t)\hat{j}=f(t)\hat{i}+\left[2\left(f(t)\right)^3-f(t)\right]\hat{j}$

$f'(t)=3\Rightarrow f(t)=3t+C$

Using $f(1)=1$, we get $1=3(1)+C\Rightarrow C=-2$

Thus, $f(t)=3t-2$ and $g(t)=2(3t-2)^3-(3t-2)$ and

$\vec{r}(t)=(3t-2)\hat{i}+\left[2(3t-2)^3-(3t-2)\right]\hat{j}$

$\vec{r}'(t)=3\hat{i}+\left[18(3t-2)^2-3\right]\hat{j}$

$\vec{r}''(t)=108(3t-2)\hat{j}$

$\left\|\vec{r}'(t)\right\|=\sqrt{3^2+\left[18(3t-2)^2-3\right]^2}$

$\vec{r}'(1)=3\hat{i}+\left[18(3-2)^2-3\right]\hat{j}=3\hat{i}+15\hat{j}$

$\vec{r}''(1)=108\hat{j}$

$\left\|\vec{r}'(1)\right\|=\sqrt{9+225}=\sqrt{234}$

$$a_{\vec{T}}=\frac{d^2s}{dt^2}=\frac{\vec{r}'(t)\cdot\vec{r}''(t)}{\left\|\vec{r}'(t)\right\|}=\frac{3+15(108)}{\sqrt{234}}=\frac{1620}{\sqrt{234}}\approx 105.90$$

$$\vec{r}'(1)\times\vec{r}''(1)=\begin{vmatrix}\hat{i} & \hat{j} & k\\ 3 & 15 & 0\\ 0 & 108 & 0\end{vmatrix}=324k$$

$$a_{\vec{N}}=K\left(\frac{ds}{dt}\right)^2=\frac{\left\|\vec{r}'(t)\times\vec{r}''(t)\right\|}{\left\|\vec{r}'(t)\right\|}=\frac{324}{\sqrt{234}}\approx 21.18$$

12) Use Theorem 15.25 to prove that if a point moves through space with an acceleration that is always $\vec{0}$, then the motion is on a line.
Proof:

$$K = \frac{\left\| \vec{r}'(t) \times \vec{r}''(t) \right\|}{\left\| \vec{r}'(t) \right\|^3} = \frac{a_{\vec{N}}}{\left\| r'(t) \right\|^2} = 0$$

$$\Rightarrow K = \left| \frac{d\theta}{ds} \right| = 0$$

$$\Rightarrow \theta = c, \text{ a constant}$$