MATH 267 CHAPTER 15 VECTOR-VALUED FUNCTIONS AND SPACE CURVES

15.4 CURVATURE

Unit Tangent Vector $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\left\|\vec{r}'(t)\right\|}$

Principal Unit Normal Vector $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}$

Exercise

(a) Find the unit tangent and normal vectors $\vec{T}(t)$ and $\vec{N}(t)$ for the curve C determined by $\vec{r}(t)$.

(b) Sketch the graph of C, and show $\vec{T}(t)$ and $\vec{N}(t)$ for the given value of t.

4) $\vec{r}(t) = (4+\cos t)\hat{i} - (3-\sin t)\hat{j};\ \ t = \dfrac{\pi}{6}$

Solution:

(a)

$$\vec{r}(t) = (4+\cos t)\hat{i} - (3-\sin t)\hat{j}$$

$$\vec{r}'(t) = -\sin t\hat{i} + \cos t\hat{j}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\left\|\vec{r}'(t)\right\|} = \frac{-\sin t\hat{i}+\cos t\hat{j}}{\sqrt{\sin^2 t+\cos^2 t}} = -\sin t\hat{i}+\cos t\hat{j}$$

$$\vec{T}'(t) = -\cos t\hat{i} - \sin t\hat{j};\quad \left\|\vec{T}'(t)\right\| = 1$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|} = -\cos t\hat{i} - \sin t\hat{j}$$

(b)

$$x = 4+\cos t,\qquad y = -3+\sin t$$

$$x-4 = \cos t,\qquad y+3 = \sin t$$

$$(x-4)^2 + (y+3)^2 = 1$$

$$\vec{T}\left(\frac{\pi}{6}\right) = \vec{r}'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6}\hat{i} + \cos\frac{\pi}{6}\hat{j} = -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$$

$$\vec{N}\left(\frac{\pi}{6}\right) = -\cos\frac{\pi}{6}\hat{i} - \sin\frac{\pi}{6}\hat{j} = -\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j}$$

6) $r(t) = t\hat{i} + \frac{1}{2}t^2\hat{j} + t^2\hat{k};\quad t = 1$

Solution:

(a)

$$\vec{r}(t) = t\hat{i} + \frac{1}{2}t^2\hat{j} + t^2\hat{k}$$

$$\vec{r}'(t) = \hat{i} + t\hat{j} + 2t\hat{k}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\left\|\vec{r}'(t)\right\|} = \frac{\hat{i} + t\hat{j} + 2t\hat{k}}{\sqrt{1 + t^2 + 4t^2}} = \frac{\hat{i} + t\hat{j} + 2t\hat{k}}{\sqrt{1 + 5t^2}}$$

$$\vec{T}'(t) = \frac{-5t\hat{i} + \hat{j} + 2\hat{k}}{\left(1 + 5t^2\right)^{3/2}}$$

$$\left\|\vec{T}'(t)\right\| = \frac{\sqrt{5}}{1 + 5t^2}$$

$$\vec{N}(t) = \frac{-5t\hat{i} + \hat{j} + 2\hat{k}}{\left(5 + 25t^2\right)^{1/2}}$$

(b)

$$\underbrace{x = t, \qquad y = \frac{1}{2}t^2,}$$

$$y = \frac{1}{2}x^2$$

$$\underbrace{x = t, \qquad z = t^2}$$

$$z = x^2$$

The curve is the intersection of the two parabolic cylinders.

$$\vec{T}(1) = \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}}, \quad \vec{N}(1) = \frac{-5\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{30}}$$

Parametrization of a Smooth Plane Curve

Definition A variable s is an **arc length parameter** for a plane curve C if there is a parametrization

$$x = f(s), \quad y = g(s)$$

such that s is the length of C from a fixed point A on the curve to the point $P(x, y)$. The parametric equations are called an **arc length parametrization** for C.

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$C: x = f(t), \quad y = g(t)$ smooth parametrization of curve C

Then the relationship between t and the arc length parameter s is given by

$$s = \int_a^t \sqrt{\left[f'(u)\right]^2 + \left[g'(u)\right]^2}\ du,$$

where u is the value of t corresponding to the fixed point A.

Use this relationship to express the given curve in terms of the arc length parameter s if the fixed point A corresponds to $t = 0$. (Hint: First evaluate the integral to find the relationship between t and s, and then substitute for t in the parametric equations.)

50) $x = 3t^2, \quad y = 2t^3; \quad t \ge 0$

Solution:

$$\begin{aligned} s &= \int_0^t \sqrt{\left[f'(u)\right]^2 + \left[g'(u)\right]^2}\ du \\ &= \int_0^t \sqrt{\left[6u\right]^2 + \left[6u^2\right]^2}\ du \\ &= \int_0^t \sqrt{36u^2 + 36u^4}\ du \\ &= \int_0^t 6u\sqrt{1+u^2}\ du \\ &= \left[2\left(1+u^2\right)^{3/2}\right]_0^t \\ &= 2\left(1+t^2\right)^{3/2} - 2 \Rightarrow t = \left[\left(\frac{s+2}{2}\right)^{2/3} - 1\right]^{1/2} \end{aligned}$$

Thus,

$$x = 3\left[\left(\frac{s+2}{2}\right)^{2/3} - 1\right], \quad y = 2\left[\left(\frac{s+2}{2}\right)^{2/3} - 1\right]^{3/2}; \quad s \ge 0.$$

What is the value of an arc length parametrization?

If a curve has an arc length parametrization then the position vector for the point $P(x, y)$ on C is

$$\vec{r}(s) = x\hat{i} + y\hat{j} = f(s)\hat{i} + g(s)\hat{j}$$

Differentiating wrt s we get a tangent vector to C,

$$\vec{r}'(s) = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} = f'(s)\hat{i} + g'(s)\hat{j}$$

and the magnitude of $\vec{r}'(s)$ is

$$\left\|\vec{r}'(s)\right\| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} = \sqrt{\left(\frac{ds}{ds}\right)^2} = 1$$

This means that for an arc length parametrization, $\vec{r}'(s)$ is also a unit tangent vector to *C* at *P*; we denote this by $\vec{T}(s)$.

For each value of *s*, let θ denote the angle between $\vec{T}(s)$ and $\hat{i}$. Note that θ is a function of *s*, since $\vec{T}(s)$ is a function of *s*. We may use the rate of change $d\theta/ds$ of θ wrt *s* to measure how much the curve *C* bends at various points.

Definition Let a smooth plane curve *C* have an arc length parametrization $x = f(s), \quad y = g(s)$, and let θ be the angle between the unit tangent vector $\vec{T}(s)$ and $\hat{i}$. The **curvature** *K* of *C* at the point $P(x, y)$ is

$$K = \left|\frac{d\theta}{ds}\right|.$$

Example 4 p 773 Prove that the curvature of a line *l* is 0 at every point on *l*.

Proof:

Let *A* be a fixed point on *l*, and let $s = d(A, P)$. Since the unit tangent vector $\vec{T}(s)$ to *l* always lies on *l*, the angle θ is the same for every *s*, thus $K = \left|\frac{d\theta}{ds}\right| = 0$.

Example 5 p 773 Prove that the curvature at every point on a circle of radius *k* is $1/k$.

Proof:

Assume the circle has its center at the origin *O*, $A(k,0)$ is the fixed point, and *P* is in Q1. Let *s* be the length of the arc $\widehat{AP}$. If α is the radian measure of angle *POA*, then

$$s = k\alpha, \quad \text{or} \quad \alpha = \frac{s}{k}.$$

Using the fact that $\vec{T}(s)$ is orthogonal to $\overrightarrow{OP}$, we have

$$\theta = \alpha + \frac{\pi}{2} = \frac{s}{k} + \frac{\pi}{2}$$
$$\frac{d\theta}{ds} = \frac{d}{ds}\left(\frac{s}{k} + \frac{\pi}{2}\right) = \frac{1}{k} + 0 = \frac{1}{k} \Rightarrow K = \left|\frac{d\theta}{ds}\right| = \frac{1}{k}$$

Note: As the radius *k* of a circle increases, its curvature *K* decreases.

$k \to \infty, \;\; K \to 0$ (curvature of a line)

$k \to 0, \;\; K \to \infty$

Formula for Finding *K*:

Theorem If a smooth curve C is the graph of $y=f(x)$, then the curvature K at $P(x,y)$ is

$$K=\frac{|y''|}{\left[1+(y')^2\right]^{3/2}}.$$

Proof:

Let $\vec{T}(s)$ and θ be defined as above. Since y' is the slope of the tangent line at P, we have

$$\tan\theta = y' \quad \text{or} \quad \theta=\tan^{-1} y' \qquad (1).$$

The arc length function may be defined by

$$s(x)=\int_a^x \sqrt{1+(y')^2}\,dx \qquad (2)$$

where a is the x-coordinate of a fixed point A on C. If y'' exists, then by the chain rule,

$$\frac{d\theta}{dx}=\frac{d\theta}{ds}\frac{ds}{dx}, \quad \text{or} \quad \frac{d\theta}{ds}=\frac{d\theta/dx}{ds/dx}$$

and by definition

$$K=\left|\frac{d\theta}{ds}\right|=\left|\frac{d\theta/dx}{ds/dx}\right|.$$

From (1) and (2)

$$\frac{d\theta}{dx}=\frac{d}{dx}\tan^{-1} y'=\frac{y''}{1+(y')^2}$$

$$\frac{ds}{dx}=\frac{d}{dx}\int_a^x \sqrt{1+(y')^2}\,dx=\sqrt{1+(y')^2}$$

Substitute these in the formula for K.

Exercises

Find the curvature of the curve at P.

8) $y = x^4; \quad P(1,1)$

Solution:

$$y = x^4$$
$$y' = 4x^3 \Rightarrow y' = 4$$
$$y'' = 12x^2 \Rightarrow y'' = 12$$
$$K = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}} = \frac{12}{\left(1+4^2\right)^{3/2}} = \frac{12}{17^{3/2}} \approx 0.17$$

Next, to derive a formula for K if C is described in terms of any parameter t:

Suppose C is given parametrically by $x = f(t), \quad y = g(t)$
and that f'' and g'' exist for every t in some interval I.

$$K = \left|\frac{d\theta}{ds}\right| = \left|\frac{d\theta/dt}{ds/dt}\right|.$$

Since the slope of the tangent line at $P(x,y)$ is $g'(t)/f'(t)$,

$$\tan\theta = \frac{g'(t)}{f'(t)}, \quad \text{or} \quad \theta = \tan^{-1}\frac{g'(t)}{f'(t)}, \quad f'(t) \neq 0 .$$

Hence

$$\frac{d\theta}{dt} = \frac{1}{1+\left[g'(t)/f'(t)\right]^2}\frac{f'(t)g''(t)-g'(t)f''(t)}{\left[f'(t)\right]^2} = \frac{f'(t)g''(t)-g'(t)f''(t)}{\left[f'(t)\right]^2+\left[g'(t)\right]^2}$$

Also,

$$\left|\frac{ds}{dt}\right| = \sqrt{\left[f'(t)\right]^2+\left[g'(t)\right]^2}$$

Substitute these in the formula for K to get:

Theorem If a plane curve C has a parametrization $x = f(t)$, $y = g(t)$ and if f'' and g'' exist, then the curvature K at $P(x,y)$ is

$$K = \frac{|f'(t)g''(t)-g'(t)f''(t)|}{\left[(f'(t))^2+(g'(t))^2\right]^{3/2}}.$$

Exercise

14) $x = t + 1, \quad y = t^2 + 4t + 3; \quad P(1,3)$

Solution:

$$x = t + 1, \quad y = t^2 + 4t + 3; \quad P(1,3)$$

$$x' = 1 \qquad y' = 2t + 4$$

$$x'' = 0 \qquad y'' = 2$$

$$\text{at } P(1,3), \quad t = x - 1 = 0$$

$$K = \frac{|(1)(2) - (4)(0)|}{\left[1^2 + 4^2\right]^{3/2}} = \frac{2}{17^{3/2}} \approx 0.03$$

Remark If $K \neq 0$, then the circle of radius $\rho = 1/K$ whose center lies on the concave side of C and has the same tangent line at P as C is the **circle of curvature** for P. Its radius ρ and center are the **radius of curvature** and **center of curvature**, respectively, for P. The circle of curvature may be thought of as the circle that best coincides with C at P.

Exercise

20) Given $y = \sec x$ and $P(0,1)$.

(a) Find the radius of curvature.
(b) Find the center of curvature.
(c) Sketch the graph and the circle of curvature for P.

Solution:

$$y = \sec x$$

$$y' = \sec x \tan x$$

$$y'' = \sec^3 x + \sec x \tan^2 x$$

$$\text{at } P(0,1)$$

$$y'(0) = \sec 0 \tan 0 = 1 \cdot 0 = 0$$

$$y''(0) = \sec^3 0 + \sec 0 \tan^2 0 = 1$$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = 1$$

$$\rho = \frac{1}{K} = 1 \quad \text{radius of curvature}$$

The center of curvature is on a vertical line 1 unit above P, i.e. at $(0,2)$.

Curvature for Space Curves

This is not developed as a simple extension of plane curves because the unit tangent vector $\vec{T}(s)$ cannot be specified in terms of a single angle θ.

Note that in two dimensions $\vec{T}(s)$ can be written as

$$\vec{T}(s) = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

where θ is the angle between $\vec{T}(s)$ and $\hat{i}$. If we consider θ as a function of *s* and we differentiate we get

$$\vec{T}'(s) = \left(-\sin\theta\frac{d\theta}{ds}\right)\hat{i} + \left(\cos\theta\frac{d\theta}{ds}\right)\hat{j} = \frac{d\theta}{ds}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)$$
$$\left\|\vec{T}'(s)\right\| = \left\|\frac{d\theta}{ds}\right\|\left\|-\sin\theta\hat{i} + \cos\theta\hat{j}\right\| = \left\|\frac{d\theta}{ds}\right\| = K$$

We will use this fact to define curvature in three dimensions.

Suppose a curve *C* has arc length parametrization

$$x = f(s), \qquad y = g(s), \qquad z = h(s)$$

where f'', g'', and h'' exist. *s* is the arc length along *C* from a fixed point *A* to the point $P\big(f(s), g(s), h(s)\big)$.

As in 2D we have

$\vec{r}(s) = f(s)\hat{i} + g(s)\hat{j} + h(s)\hat{k}$ position vector for *P*

$\vec{r}'(s) = \vec{T}(s)$ unit tangent vector to *C* at *P*

Since $\left\|\vec{T}(s)\right\|$ is a constant, it follows that $\vec{T}'(s)$ is ortogonal to $\vec{T}(s)$.

$\vec{N}(s) = \dfrac{\vec{T}'(s)}{\left\|\vec{T}'(s)\right\|}$ is a unit vector ortogonal to $\vec{T}(s)$ and is called the **principal unit normal vector** to *C* at point *P*.

Definition Let a smooth space curve *C* have an arc length parametrization

$$x = f(s), \quad y = g(s), \quad z = h(s).$$

Let $\vec{r}(s) = f(s)\vec{i} + g(s)\vec{j} + h(s)\vec{k}$ and let $\vec{T}(s) = \vec{r}'(s)$. The **curvature** *K* of *C* at the point $P(x, y, z)$ is

$$K = \left\|\vec{T}'(s)\right\|.$$

Note: The preceding definition reduces to $K = \left|\dfrac{d\theta}{ds}\right|$ if *C* is a plane curve.

Also,

$$\vec{N}(s) = \frac{1}{K}\vec{T}'(s) \Leftrightarrow \vec{T}'(s) = K\vec{N}(s).$$

Exercises

Find the points on the given curve at which the curvature is a maximum.

24) $y = \cosh x$

Solution:

$$y = \cosh x$$
$$y' = \sinh x$$
$$y'' = \cosh x$$
$$K(x) = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\cosh x}{\left(1 + \sinh^2 x\right)^{3/2}} = \frac{1}{\cosh^2 x}$$

$K(x)$ is a maximum when $\cosh x$ is minimum. i.e. when $x = 0$.

Thus, $(0,1)$ is the point of maximum curvature.

Find the points on the graph at which the curvature is 0.

32) $y = e^{-x^2}$

Solution:

$$y = e^{-x^2}$$
$$y' = -2xe^{-x^2}$$
$$y'' = -2e^{-x^2} + 4x^2 e^{-x^2}$$
$$K(x) = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{-2e^{-x^2} + 4x^2 e^{-x^2}}{\left[1 + \left(-2xe^{-x^2}\right)^2\right]^{3/2}} = 0$$
$$\Rightarrow -2e^{-x^2} + 4x^2 e^{-x^2} = 0 \Rightarrow 4x^2 = 2 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

Curvature for Polar Curves

33) Suppose a curve C is the graph of a polar equation $r = f(\theta)$. If $r' = dr/d\theta$ and $r'' = d^2r/d\theta^2$ then

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}}.$$

34) Use the formula in #33 to find the curvature at $P(r,\theta)$ of $r = a(1 - \cos\theta)$; $0 < \theta < 2\pi$.

Solution:

$$r = a(1 - \cos\theta)$$
$$r' = a\sin\theta$$
$$r'' = a\cos\theta$$

$$K=\frac{\left|2(r')^2-rr''+r^2\right|}{\left[(r')^2+r^2\right]^{3/2}}$$

$$=\frac{\left|2a^2\sin^2\theta-a^2\cos\theta(1-\cos\theta)+a^2(1-\cos\theta)^2\right|}{\left[a^2\sin^2\theta+a^2(1-\cos\theta)^2\right]^{3/2}}$$

$$=\frac{\left|2a^2\sin^2\theta-a^2\cos\theta+a^2\cos^2\theta+a^2\left(1-2\cos\theta+\cos^2\theta\right)\right|}{\left[a^2\sin^2\theta+a^2\left(1-2\cos\theta+\cos^2\theta\right)\right]^{3/2}}$$

$$=\frac{\left|3a^2-3a^2\cos\theta\right|}{\left[2a^2-2a^2\cos\theta\right]^{3/2}}$$

$$=\frac{3a^2\left|1-\cos\theta\right|}{\left(2a^2\right)^{3/2}(1-\cos\theta)^{3/2}}=\frac{3}{2\sqrt{2}a(1-\cos\theta)^{1/2}}$$

Formula for the Center of Curvature

37) Let $P(x,y)$ be a point on the graph of $y=f(x)$ at which $K\neq 0$. If (h,k) is the center pf curvature for *P*, show that

$$h=x-\frac{y'\left[1+(y')^2\right]}{y''},\quad k=y+\frac{\left[1+(y')^2\right]}{y''}$$

38) Use #37 to find the center of curvature for the point $P(1,1)$ on the graph of $y=2-x^3$.

Solution:

$$y=2-x^3$$

$$y'=-3x^2;\ \ y'(1)=-3(1)^2=-3$$

$$y''=-6x;\ \ y''(1)=-6(1)=-6$$

$$h=x-\frac{y'\left[1+(y')^2\right]}{y''},\quad k=y+\frac{y'\left[1+(y')^2\right]}{y''}$$

$$h=1-\frac{-3\left[1+(-3)^2\right]}{-6},\quad k=1+\frac{-3\left[1+(-3)^2\right]}{-6}$$

$$=1-5\quad,\quad =1+5$$

$$=-4\quad,\quad =6$$

$$(h,k)=(-4,6)$$