MATH 267 CHAPTER 15 VECTOR-VALUED FUNCTIONS AND SPACE CURVES

15.3 MOTION

We will study the motion of a point *P* in a coordinate plane. For objects we will assume that the mass is concentrated at *P*. Let the coordinates of *P* be given by: $x = f(t), \quad y = g(t), \quad t \in I$.

Let $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j}$. Then as *t* varies through *I*, the endpoint of $\vec{r}(t)$ traces the path *C* of the point. We call $\vec{r}(t)$ the **position vector** of *P,* and

$$\vec{r}\,'(t) = f'(t)\hat{i} + g'(t)\hat{j}$$

is the tangent vector to *C* with initial point *P*. The vector $\vec{r}\,'(t)$ points in the direction of increasing values of *t* and has magnitude

$$\left\|\vec{r}\,'(t)\right\| = \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\,.$$

Let t_0 be any number in *I*, and let P_0 be the point on *C* that corresponds to t_0. If *C* is smooth, then the arc length $s(t)$ of *C* from P_0 to *P* is

$$s(t) = \int_{t_0}^{t} \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\,dt = \int_{t_0}^{t} \left\|\vec{r}\,'(t)\right\|\,dt\,.$$

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$$D_t\left[s(t)\right] = D_t\int_{t_0}^{t} \left\|\vec{r}\,'(t)\right\|\,dt = \left\|\vec{r}\,'(t)\right\|$$

and thus $\left\|\vec{r}\,'(t)\right\|$ is the *rate of change of arc length with respect to time* and we refer to $\left\|\vec{r}\,'(t)\right\|$ as the **speed** of the point. The tangent vector $\vec{r}\,'(t)$ is defined as the **velocity** of the point *P* at time *t*, and the vector $\vec{r}\,''(t)$ is the **acceleration** of *P*. $\vec{r}\,''(t)$ will also be represented by a vector with initial point at *P* and in most cases will be directed toward the concave side of *C*.

The following summarizes the above discussion:

Definition Let the **position vector** for a point $P(x,y)$ moving in an *xy* – plane be

$$\vec{r}(t) = x\vec{i} + y\vec{j} = f(t)\vec{i} + g(t)\vec{j}\,,$$

where *t* is time and *f* and *g* have first and second derivatives. Then

velocity $\quad \vec{v}(t) = \vec{r}\,'(t) = \dfrac{dx}{dt}\vec{i} + \dfrac{dy}{dt}\vec{j}$

speed $\quad v(t) = \left\|\vec{v}(t)\right\| = \left\|\vec{r}\,'(t)\right\| = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$

acceleration $\quad \vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \dfrac{d^2x}{dt^2}\vec{i} + \dfrac{d^2y}{dt^2}\vec{j}$

Exercise

Let $\vec{r}(t)$ be the position vector of a moving point *P*. Sketch the path *C* of *P* together with $\vec{v}(t)$ and $\vec{a}(t)$ for the given value of *t*.

2) $\vec{r}(t)=\left(4-9t^2\right)\hat{i}+3t\ \hat{j};\ \ t=1$

Solution:

$$\vec{r}(t)=\left(4-9t^2\right)\hat{i}+3t\ \hat{j};\ \vec{r}(1)=\left(4-9(1)^2\right)\hat{i}+3(1)\hat{j}=-5\hat{i}+3\hat{j}$$

$$\vec{v}(t)=\vec{r}\,'(t)-18t\ \hat{i}+3\hat{j};\ \vec{v}(1)=-18(1)\ \hat{i}+3\hat{j}=-18\hat{i}+3\hat{j}$$

$$\vec{a}(t)=\vec{r}\,''(t)=-18\ \hat{i};\ \vec{a}(1)=-18\ \hat{i}$$

$$x=4-9t^2, \qquad y=3t$$

$x=4-y^2$ parabola with vertex at $(4,0)$ and *y*-intercepts ± 2

4) $\vec{r}(t)=\cos^2 t\ \hat{i}+2\sin t\ \hat{j};\ \ t=\frac{3\pi}{4}$

Solution:

$$\vec{r}(t)=\cos^2 t\ \hat{i}+2\sin t\ \hat{j};$$

$$\vec{v}(t)=-2\cos t\sin t\ \hat{i}+2\cos t\ \hat{j}=-\sin 2t\ \hat{i}+2\cos t\ \hat{j};$$

$$\vec{a}(t)=-2\sin 2t\cos 2t\ \hat{i}-2\sin t\ \hat{j}=-\sin 4t\ \hat{i}-2\sin t\ \hat{j};$$

$$\vec{r}\left(\frac{3\pi}{4}\right)=\cos^2\left(\frac{3\pi}{4}\right)\ \hat{i}+2\sin\left(\frac{3\pi}{4}\right)\ \hat{j}=\frac{1}{2}\ \hat{i}+\sqrt{2}\ \hat{j}$$

$$\vec{v}\left(\frac{3\pi}{4}\right)=-\sin 2\left(\frac{3\pi}{4}\right)\ \hat{i}+2\cos\left(\frac{3\pi}{4}\right)\ \hat{j}=\hat{i}-\sqrt{2}\ \hat{j}$$

$$\vec{a}\left(\frac{3\pi}{4}\right)=-\sin 4\left(\frac{3\pi}{4}\right)\ \hat{i}-2\sin\left(\frac{3\pi}{4}\right)\ \hat{j}=-\sqrt{2}\ \hat{j}$$

$$x=\cos^2 t\ , \qquad y=2\sin t$$

$$x=1-\sin^2 t, \qquad \frac{y}{2}=\sin t$$

$x=1-\frac{y^2}{4}$ parabola with vertex at $(1,0)$ and $y-$intercepts ± 2

If $\vec{r}(t)$ is the position vector of a moving point *P*, find its velocity, acceleration, and speed at the given time *t*.

10) $\vec{r}(t)=\sqrt{t}\ \hat{i}+\left(1+\sqrt{t}\right)\hat{j};\ \ t=4$

Solution:

$$\vec{r}(t)=\sqrt{t}\ \hat{i}+\left(1+\sqrt{t}\right)\hat{j}$$

$$\vec{v}(t)=\frac{1}{2\sqrt{t}}\ \hat{i}+\ \frac{1}{2\sqrt{t}}\hat{j};\ \ \vec{v}(4)=\frac{1}{2\sqrt{4}}\ \hat{i}+\ \frac{1}{2\sqrt{4}}\hat{j}=\frac{1}{4}\ \hat{i}+\ \frac{1}{4}\hat{j}$$

$$\vec{a}(t)=-\frac{1}{4t^{3/2}}\ \hat{i}-\frac{1}{4t^{3/2}}\hat{j};\ \vec{a}(4)=-\frac{1}{4(4)^{3/2}}\ \hat{i}-\frac{1}{4(4)^{3/2}}\hat{j}=-\frac{1}{32}\ \hat{i}-\frac{1}{32}\hat{j}$$

$$\left\|\vec{v}(4)\right\|=\sqrt{\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2}=\frac{1}{2\sqrt{2}}$$

Uniform Circular Motion

Given: a point *P* moving around a circle of radius *k* at a constant speed *v*
Show: the acceleration vector has constant magnitude v^2/k and is directed from *P* toward the center of the circle.
Solution: Assume the center of the circle is at the origin *O* in an *xy*-plane and that the orientation is counterclockwise. Suppose at time $t = 0$ the point is at $A(k,0)$ and that θ is the angle generated by $\overrightarrow{OP}$ after *t* units of time. Since *P* moves around the circle at a constant speed, the rate of change of θ with respect to *t* (the **angular speed**) is a constant ω:

$$\frac{d\theta}{dt} = \omega \text{ or } d\theta = \omega\, dt .$$

Integrating, we get

$$\theta = \omega t + c$$

for some constant *c*. Since $\theta = 0$ when $t = 0$, we get $c = 0$, i.e.

$$\theta = \omega t .$$

Thus, the coordinates of *P* are

$$x = k\cos\omega t, \quad y = k\sin\omega t,$$

and the position vector of *P* is

$$\vec{r}(t) = k\cos\omega t\,\hat{i} + k\sin\omega t\,\hat{j} ,$$

and

$$\vec{v}(t) = -\omega k\sin\omega t\,\hat{i} + \omega k\cos\omega t\,\hat{j}$$

$$\vec{a}(t) = -\omega^2 k\cos\omega t\,\hat{i} - \omega^2 k\sin\omega t\,\hat{j}$$

$$= -\omega^2\left(k\cos\omega t\,\hat{i} + k\sin\omega t\,\hat{j}\right) = -\omega^2\vec{r}(t)$$

This shows that the direction of the acceleration vector is *opposite* that of $\vec{r}(t)$, and hence $\vec{a}(t)$ is directed from *P* toward *O*. Also

$$\left\|\vec{a}(t)\right\| = \left\|-\omega^2\vec{r}(t)\right\| = \left|-\omega^2\right|\left\|\vec{r}(t)\right\| = \omega^2 k, \text{ a constant.}$$

Moreover,

$$\upsilon = \left\|\vec{r}\,'(t)\right\| = \sqrt{(-\omega k)^2\sin^2\omega t + (\omega k)^2\cos^2\omega t} = \sqrt{\omega^2 k^2} = \omega k$$

and thus

$$\upsilon = \omega k.$$

Substituting this in the formula $\left\|\vec{a}(t)\right\| = \omega^2 k$ gives

$$\left\|\vec{a}(t)\right\| = \frac{\upsilon^2}{k} .$$

The acceleration vector is called the **centripetal acceleration vector**, and the force that produces $\vec{a}(t)$ is called the **centripetal force**.

Note: The magnitude $\left\|\vec{a}(t)\right\| = \frac{\upsilon^2}{k}$ will increase if we either increase υ or decrease *k*.

To summarize the above discussion, we have:

Motion around a circle of radius *k* at a constant speed *v*

$\upsilon = \omega k$ speed is the product of the radius and the angular speed

$\left\|\vec{a}(t)\right\| = \frac{\upsilon^2}{k}$ the magnitude of acceleration is the ratio of the square of the speed and the radius

$T = \frac{2\pi}{\omega}$ time required for one revolution

Newton's Second Law of Motion

The force $\vec{F}$ acting on an object of constant mass *m* is related to the acceleration $\vec{a}$ of the object as follows:

$$\vec{F} = m\vec{a}$$

Example In the circular motion described above, the force acting on an object of mass *m* is given by

$$\vec{F}(t) = m\vec{a}(t) = -m\omega^2\vec{r}(t)$$

and points toward the origin.

Exercise

20) An earth satellite is in circular orbit. If the time required for one revolution is 88 minutes, approximate the satellite's altitude. Assume the radius of the earth is 4000 miles.

Solution:

Convert 88 minutes to 88*60 seconds and 32 to 32/5280 mile

$$T = \frac{2\pi}{\omega} \Rightarrow 88\cdot 60 = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{88\cdot 60} = \frac{2\pi}{5280}$$

$$\left\|\vec{a}(t)\right\| = k\omega^2$$

$$\frac{32}{5280} = k\left(\frac{2\pi}{5280}\right)^2$$

$$k = \frac{32(5280)^2}{5280\cdot 4\pi^2} \approx 4280 \text{ mi}$$

Thus, the altitude is approximately 280 miles.

Projectile Motion

Given: A projectile is fired with an initial velocity $\vec{v}_0$ at an angle of elevation α from a point h_0 feet (meters) above the ground; the only force acting is gravity $\vec{g}$.

Find:

(a) the position vector $\vec{r}(t)$ of the projectile

For the following, assume the projectile started from the ground:

(b) the range of the projectile (assuming the projectile is fired from the ground)
(c) the maximum range of the projectile (assuming the projectile is fired from the ground)
(d) the maximum altitude of the projectile

Solution:

(a)

$$\vec{g} = -g\,\hat{j},\quad \left\|\vec{g}\right\| = 32 \text{ ft/sec}^2 \ \ (9.8 \text{ m/sec}^2)$$

$$\vec{F} = m\vec{a} = -m\vec{g} \Leftrightarrow \vec{a} = -g\,\hat{j}$$

$$\vec{r}''(t) = -g\,\hat{j}$$

$$\vec{r}'(t) = -g\,t\,\hat{j} + \vec{c} \text{ for some constant vector } \vec{c}$$

$$\vec{v}_0 = \vec{r}'(0) = \vec{c}$$

$$\vec{r}'(t) = -gt\,\hat{j} + \vec{v}_0$$

$$\vec{r}(t) = -\frac{1}{2}gt^2\,\hat{j} + \vec{v}_0 t + \vec{d},\ \text{ for some constant vector } \vec{d}$$

$$\vec{r}(0) = h_0\,\hat{j} \Rightarrow \vec{d} = h_0\,\hat{j}$$

$$\vec{r}(t) = -\frac{1}{2}gt^2\,\hat{j} + \vec{v}_0 t + h_0\,\hat{j} = \left(-\frac{1}{2}t^2 g + h_0\right)\hat{j} + \vec{v}_0 t$$

Now, if we let $v_0 = \left\|\vec{v}_0\right\|$ (the initial speed of the projectile), then

$$\vec{v}_0 = v_0\cos\alpha\,\hat{i} + v_0\sin\alpha\,\hat{j}$$

and we can write the position vector as

$$\vec{r}(t) = \left(-\frac{1}{2}t^2 g + h_0\right)\hat{j} + \left(v_0\cos\alpha\,\hat{i} + v_0\sin\alpha\,\hat{j}\right)t$$

$$\vec{r}(t) = (v_0\cos\alpha)t\,\hat{i} + \left[(v_0\sin\alpha)t - \frac{1}{2}t^2 g + h_0\right]\hat{j}$$

This gives us parametric equations for the trajectory:

$$x = (v_0\cos\alpha)t, \qquad y = (v_0\sin\alpha)t - \frac{1}{2}t^2 g + h_0$$

For (b) – (d), let us assume $h_0 = 0$.

(b) The range is the horizontal distance and is the value of *x* when *y* is 0. So,

$$y = (v_0\sin\alpha)t - \frac{1}{2}t^2 g = 0 \Leftrightarrow t = 0 \text{ or } t = \frac{2v_0\sin\alpha}{g}.$$

The second value of *t* will give the range *d* or the horizontal distance travelled:

$$d = (v_0\cos\alpha)\frac{2v_0\sin\alpha}{g} = \frac{2v_0^2\cos\alpha\sin\alpha}{g} = \frac{v_0^2\sin 2\alpha}{g}.$$

(c) Clearly, the maximum range occurs when $\sin 2\alpha = 1$ or when $\alpha = \dfrac{\pi}{4}$.

(d)The maximum altitude will occur when the tangent vector to the path is horizontal, i.e., when the $\hat{j}$ -component is 0. First, let us get the tangent vector which is the velocity vector or the derivative of the position vector:

$$\vec{r}'(t) = (v_0 \cos\alpha)\hat{i} + \left[(v_0 \sin\alpha) - gt\right]\hat{j}$$

We want $(v_0 \sin\alpha) - gt = 0$ or $t = \dfrac{v_0 \sin\alpha}{g}$. Substitute this in the parametric equation for *y* to get the maximum altitude:

$$h = (v_0 \sin\alpha)\frac{v_0 \sin\alpha}{g} - \frac{1}{2}\left(\frac{v_0 \sin\alpha}{g}\right)^2 g = \frac{v_0^2 \sin^2\alpha}{2g}$$

Exercises

22) A projectile is fired from level ground with an initial speed of 1500 ft/sec and angle of elevation of 60^0. Find

a) the velocity at time *t*

b) the maximum altitude

c) the range

d) the speed at which the projectile strikes the ground

Solution:

a)

$$\vec{a}(t) = -32\hat{j} \Rightarrow \vec{v}(t) = -32t\,\hat{j} + \vec{v}_0 = -32t\,\hat{j} + 1500\left(\cos 60^\circ\,\hat{i} + \sin 60^\circ\,\hat{j}\right)$$

$$\vec{v}(t) = 750\hat{i} + \left(750\sqrt{3} - 32t\right)\hat{j} \quad \text{velocity vector at time } t$$

$$\vec{r}(t) = 750t\,\hat{i} + \left(750t\sqrt{3} - 16t^2\right)\hat{j} \quad \text{position vector at time } t$$

b) Set the $\hat{j}$ component of the velocity vector to 0:

$$750\sqrt{3} - 32t = 0 \Leftrightarrow t = \frac{750\sqrt{3}}{32}$$

Substitute in the $\hat{j}$ component of the position vector:

$$h = 750\sqrt{3}\left(\frac{750\sqrt{3}}{32}\right) - 16\left(\frac{750\sqrt{3}}{32}\right)^2 \approx 26{,}367 \text{ ft}$$

c) Set the $\hat{j}$ component of the position vector to 0:

$$750t\sqrt{3} - 16t^2 = 0 \Leftrightarrow t = 0 \text{ or } t = \frac{750\sqrt{3}}{16}$$

Substitute the second value of *t* to the $\hat{i}$ component of the position vector:

$$d = 750\left(\frac{750\sqrt{3}}{16}\right) \approx 60{,}892 \text{ ft}$$

d) Substitute $t = \dfrac{750\sqrt{3}}{16}$ in the velocity equation and get the magnitude:

$$\vec{v}(t) = 750\hat{i} + \left(750\sqrt{3} - 32\cdot\frac{750\sqrt{3}}{16}\right)\hat{j} = 750\hat{i} - 750\sqrt{3}\,\hat{j}$$

$$\left\|\vec{v}(t)\right\| = \sqrt{(750)^2 + \left(-750\sqrt{3}\right)^2} = 1500 \text{ ft/sec}$$

24) A projectile is fired horizontally with a velocity of 1800 ft/sec from an altitude of 1000 feet above level ground. When and where does it strike the ground?
Solution:
Given: $\vec{v}_0 = 1800\hat{i}$

$h_0 = 1000$ ft

$$\vec{r}(t) = \left(-\frac{1}{2}t^2 g + h_0\right)\hat{j} + \vec{v}_0 t$$
$$\vec{r}(t) = \left(-16t^2 + 1000\right)\hat{j} + 1800t\,\hat{i}$$

When the projectile strikes the ground we have $y = 0$ or

$$-16t^2 + 1000 = 0$$
$$t^2 = \frac{1000}{16} \Rightarrow t \approx 7.9 \text{ sec}$$

and $\vec{r}(7.9) = 1800(7.9)\hat{i} \approx 14{,}230$ ft from a point directl y under the firing position

Additional Exercises

1) The position function of a particle is given by $\vec{r}(t) = \left\langle t^2, 5t, t^2 - 16t \right\rangle$. When is the speed a minimum?

2) A projectile is fired with an initial speed of 200 m/s and an angle of elevation of 60^0. Find
a) the range of the projectile,
b) the maximum height reached, and
c) the speed at impact.

3) A ball is thrown at an angle of 45^0. If the ball lands 90 m away, what was the initial speed of the ball?

4) A gun is fired with angle of elevation of 30^0. What is the muzzle speed if the maximum height of the shell is 500 m?