MATH 267 CHAPTER 15 VECTOR-VALUED FUNCTIONS AND SPACE CURVES

15.2 LIMITS, DERIVATIVES, AND INTEGRALS

Definition Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$. The **<u>limit</u> of** $\vec{r}(t)$ **as** t **approaches** a is

$$\lim_{t\to a}\vec{r}(t) = \left[\lim_{t\to a} f(t)\right]\vec{i} + \left[\lim_{t\to a} g(t)\right]\vec{j} + \left[\lim_{t\to a} h(t)\right]\vec{k} \ ,$$

provided *f, g,* and *h* have limits as *t* approaches *a*.

Definition A vector-valued function $\vec{r}$ is **<u>continuous</u>** at *a* if $\lim_{t\to a}\vec{r}(t) = \vec{r}(a)$.

Remark It follows that if $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$, then $\vec{r}$ is continuous at *a* if and only if *f, g,* and *h* are continuous at *a*. Continuity on an interval is defined in the usual way.

Definition Let $\vec{r}$ be a vector-valued function. The **<u>derivative</u> of** $\vec{r}$ is the vector-valued function $\vec{r}\,'$ defined by

$$\vec{r}\,'(t) = \lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

for every *t* such that the limit exists.

Geometric Interpretation The derivative $\vec{r}\,'(t)$ is the **<u>tangent vector</u>** to *C* at *P.*

Theorem If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ and *f, g,* and *h* are differentiable, then

$$\vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

If $\vec{r}\,'(t)$ exists, we say that $\vec{r}$ is **<u>differentiable</u>** at *t*. Other notations for the derivative are

$$\vec{r}\,'(t) = D_t\vec{r}(t) = \frac{d}{dt}\vec{r}(t) \ .$$

Higher derivatives may be obtained in a similar fashion.

Exercises

a) Find the domain of $\vec{r}$.

b) Find $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$

2) $\vec{r}(t) = \frac{1}{t}\hat{i} + \sin 3t\ \hat{j}$

Solution:

a) $t \neq 0$

b) $\vec{r}\,'(t) = -\frac{1}{t^2}\hat{i} + 3\cos 3t\ \hat{j}$

$\vec{r}\,''(t) = \frac{2}{t^3}\hat{i} - 9\sin 3t\ \hat{j}$

4) $\vec{r}(t) = e^{t^2}\hat{i} + \sin^{-1} t\ \hat{j}$

Solution:

a) $[-1,1]$

b) $\vec{r}'(t) = 2te^{t^2}\hat{i} + \dfrac{1}{\sqrt{1-t^2}}\ \hat{j}$

$$\vec{r}''(t) = \left(4t^2e^{t^2} + 2e^{t^2}\right)\hat{i} + \frac{t}{\left(1-t^2\right)^{\frac{3}{2}}}\ \hat{j}$$

(a) Sketch the curve in the *xy*-plane determined by $\vec{r}(t)$ and indicate the orientation.

(b) Find $\vec{r}'(t)$ and sketch $\vec{r}(t)$ and $\vec{r}'(t)$ for the indicated value of *t*.

10) $\vec{r}(t) = e^{2t}\hat{i} + e^{-4t}\hat{j};\ \ t = 0$

Solution:

$$\vec{r}(t) = e^{2t}\hat{i} + e^{-4t}\hat{j}$$

$$x = e^{2t},\quad y = e^{-4t} \Rightarrow y = x^{-2} = \frac{1}{x^2},\ x > 0$$

$$\vec{r}'(t) = 2e^{2t}\hat{i} - 4e^{-4t}\hat{j}$$

$$\vec{r}(0) = \hat{i} + \hat{j}$$

$$\vec{r}'(0) = 2\hat{i} - 4\hat{j}$$

12) $\vec{r}(t) = 2\sec t\ \hat{i} + 3\tan t\ \hat{j};\ t = \dfrac{\pi}{4},\ |t| < \dfrac{\pi}{2}$

Solution:

$$\vec{r}(t) = 2\sec t\ \hat{i} + 3\tan t\ \hat{j}$$

$$x = 2\sec t,\quad y = 3\tan t \Rightarrow \frac{x}{2} = \sec t,\quad \frac{y}{3} = \tan t$$

Using $\sec^2 t - \tan^2 t = 1$, we get $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$, and since $|t| < \dfrac{\pi}{2}$, $x = 2\sec t > 0$ and thus we need only the right branch of the hyperbola. The orientation is from down to up.

$$\vec{r}'(t) = 2\sec t\tan t\ \hat{i} + 3\sec^2 t\ \hat{j}$$

At $t = \dfrac{\pi}{4}$,

$$\vec{r}\left(\frac{\pi}{4}\right) = 2\sec\frac{\pi}{4}\ \hat{i} + 3\tan\frac{\pi}{4}\ \hat{j} = \left\langle 2\sqrt{2}, 3\right\rangle$$

$$\vec{r}'\left(\frac{\pi}{4}\right) = 2\sec\frac{\pi}{4}\tan\frac{\pi}{4}\ \hat{i} + 3\sec^2\frac{\pi}{4}\ \hat{j} = \left\langle 2\sqrt{2}, 6\right\rangle$$

Differentiation Formulas

Theorem If $\vec{u}$ and $\vec{v}$ are differentiable vector-valued functions and *c* is a scalar, then

(i) $D_t\left[\vec{u}(t)+\vec{v}(t)\right]=\vec{u'}(t)+\vec{v'}(t)$

(ii) $D_t\left[c\,\vec{u}(t)\right]=c\,\vec{u'}(t)$

(iii) $D_t\left[\vec{u}(t)\cdot\vec{v}(t)\right]=\vec{u}(t)\cdot\vec{v'}(t)+\vec{u'}(t)\cdot\vec{v}(t)$

(iv) $D_t\left[\vec{u}(t)\times\vec{v}(t)\right]=\vec{u}(t)\times\vec{v'}(t)+\vec{u'}(t)\times\vec{v}(t)$

Proof of (iii)

Let $\vec{u}(t)=\langle f_1(t),f_2(t),f_3(t)\rangle$ and $\vec{v}(t)=\langle g_1(t),g_2(t),g_3(t)\rangle$.

Then $\vec{u}(t)\bullet\vec{v}(t)=f_1(t)g_1(t)+f_2(t)g_2(t)+f_3(t)g_3(t)=\sum_{k=1}^{3}f_k(t)g_k(t)$

$$D_t\left[\vec{u}(t)\bullet\vec{v}(t)\right]=D_t\sum_{k=1}^{3}f_k(t)g_k(t)=\sum_{k=1}^{3}D_t\left[f_k(t)g_k(t)\right]$$

$$=\sum_{k=1}^{3}\left[f_k'(t)g_k(t)+\sum_{k=1}^{3}f_k(t)g_k'(t)\right]=\sum_{k=1}^{3}f_k'(t)g_k(t)+\sum_{k=1}^{3}f_k(t)g_k'(t)$$

$$=\vec{u}'(t)\bullet\vec{v}(t)+\vec{u}(t)\bullet\vec{v}'(t)$$

Exercises

18) Given $C:\ x=4\sqrt{t},\quad y=t^2-10,\quad z=4/t;\quad P(8,6,1)$ Find parametric equations for the tangent line to *C* at *P.*

Solution:

$C:\ x=4\sqrt{t},\quad y=t^2-10,\quad z=4/t;\quad P(8,6,1)$

$x'=\dfrac{2}{\sqrt{t}},\quad y'=2t,\quad z'=-\dfrac{4}{t^2}$

Find $t: x=4\sqrt{t}=8\Rightarrow t=4$

Thus,

$x'=\dfrac{2}{\sqrt{4}}=1,\quad y'=2(4)=8,\quad z'=-\dfrac{4}{4^2}=-\dfrac{1}{4}$

and a vector parallel to the tangent line is $\left\langle 1,8,-\dfrac{1}{4}\right\rangle$ and

the paramteric equations are

$x=8+t,\quad y=6+8t,\quad z=1-\dfrac{1}{4}t.$

20) $C: \quad x = t\sin t, \quad y = t\cos t, \quad z = t; \quad P\left(\frac{\pi}{2}, 0, \frac{\pi}{2}\right)$ Find parametric equations for the tangent line to C at P.

Solution:

$$C: \quad x = t\sin t, \qquad y = t\cos t, \qquad z = t; \quad P\left(\frac{\pi}{2}, 0, \frac{\pi}{2}\right)$$

$$x' = \sin t + t\cos t, \qquad y' = \cos t - t\sin t, \qquad z' = 1$$

Solve for t :

Since $z = t,$ then at $P\left(\frac{\pi}{2}, 0, \frac{\pi}{2}\right),\ t = \frac{\pi}{2}$

and

$$x' = \sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2} = 1, \qquad y' = \cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2} = -\frac{\pi}{2}.$$

Thus, parametric equations of the tangent line are

$$x = \frac{\pi}{2} + t, \qquad y = -\frac{\pi}{2}t, \qquad z = \frac{\pi}{2} + t$$

Theorem If $\vec{r}$ is differentiable and $\left\|\vec{r}(t)\right\|$ is constant, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for every t in the domain of $\vec{r}'$.

Proof:

If $\left\|\vec{r}(t)\right\| = c$ for some constant c, then

$$\vec{r}(t)\cdot\vec{r}(t) = \left\|\vec{r}(t)\right\|^2 = c^2$$

and

$$D_t\left[\vec{r}(t)\cdot\vec{r}(t)\right] = D_t c^2 = 0$$

$$\Leftrightarrow \vec{r}(t)\cdot\vec{r}'(t) + \vec{r}'(t)\cdot\vec{r}(t) = 0$$

$$\Leftrightarrow 2\left[\vec{r}(t)\cdot\vec{r}'(t)\right] = 0$$

Thus, $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal.

Note: $\left\|\vec{r}(t)\right\|$ is constant c implies that the curve must lie on the sphere with radius c and center at the origin.

22) Given: $C: \quad x = \sin t + 2, \quad y = \cos t, \quad z = t; \quad P(2,1,0)$ Find two unit tangent vectors to C at P.
Solution:

$C: \quad x = \sin t + 2, \quad y = \cos t, \quad z = t; \quad P(2,1,0)$

$x' = \cos t, \quad y' = -\sin t, \quad z' = 1$

Solve for t: $z = t \Rightarrow t = 0$ at $P(2,1,0)$

Substitute:

$x' = \cos 0 = 1, \quad y' = -\sin 0 = 0, \quad z' = 1$

Thus, a tangent vector is $\vec{r}\,'(0) = \langle 1,0,1 \rangle$ and $\left\| \vec{r}\,'(0) \right\| = \sqrt{2}$

and two unit tangent vectors are $\pm \dfrac{\langle 1,0,1 \rangle}{\sqrt{2}}$.

24) The **general helix** is the a curve whose tangent vector makes a constant angle with a fixed unit vector $\vec{u}$. Show that the curve with parametrization

$$x = 3t - t^3, \quad y = 3t^2, \quad z = 3t + t^3; \quad t \text{ in } \mathbb{R}$$

is a general helix by finding an appropriate vector $\vec{u}$.
Solution:

$x = 3t - t^3, \quad y = 3t^2, \quad z = 3t + t^3; \quad t \text{ in } \mathbb{R}$

$x' = 3 - 3t^2, \quad y' = 6t, \quad z' = 3 + 3t^2$

$\vec{r}\,'(t) = \left\langle 3 - 3t^2, 6t, 3 + 3t^2 \right\rangle$

$\left\| \vec{r}\,'(t) \right\| = \sqrt{\left(3 - 3t^2\right)^2 + (6t)^2 + \left(3t^2\right)^2} = \sqrt{18}\left(t^2 + 1\right)$

Since $\cos\theta = \dfrac{\vec{r}\,'(t) \cdot \vec{u}}{\left\| \vec{r}\,'(t) \right\|}$, then if $\vec{u} = k$, then $\cos\theta = \dfrac{3\left(1 + t^2\right)}{\sqrt{18}\left(t^2 + 1\right)} = \dfrac{3}{\sqrt{18}}$, a constant.

Definition Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$, where *f, g,* and *h* are integrable on $[a,b]$. The <u>**definite integral of**</u> $\vec{r}$ **from** a **to** b is

$$\int_a^b r(t)dt = \left[\int_a^b f(t)dt\right]\vec{i} + \left[\int_a^b g(t)dt\right]\vec{j} + \left[\int_a^b h(t)dt\right]\vec{k}.$$

If $\vec{R}'(t) = \vec{r}(t)$, then $\vec{R}(t)$ is an <u>**antiderivative**</u> of $\vec{r}(t)$.

Theorem If $\vec{R}(t)$ is an **antiderivative** of $\vec{r}(t)$, then

$$\int_a^b \vec{r}(t)dt = \vec{R}(t)\Big]_a^b = \vec{R}(b) - \vec{R}(a).$$

Note: The notion of an indefinite extends similarly. Thus,

$$\int \vec{r}(t)dt = \vec{R}(t) + \vec{C}, \text{ where } \vec{R}'(t) = \vec{r}(t).$$

Evaluate the integral.

30) $\int_0^1 \left[te^{t^2}\ \hat{i} + \sqrt{t}\ \hat{j} + \left(t^2+1\right)^{-1}\ k \right]$

Solution:

$$\int_0^1 \left[te^{t^2}\ \hat{i} + \sqrt{t}\ \hat{j} + \left(t^2+1\right)^{-1}\ k \right]$$

$$= \left[\frac{1}{2}e^{t^2}\hat{i} + \frac{2}{3}t^{\frac{3}{2}}\ \hat{j} + \tan^{-1}t\ k \right]_0^1$$

$$= \frac{1}{2}e^{1}\hat{i} + \frac{2}{3}(1)^{\frac{3}{2}}\ \hat{j} + \tan^{-1}(1)\ k - \left(\frac{1}{2}e^{0}\hat{i} + \frac{2}{3}(0)^{\frac{3}{2}}\ \hat{j} + \tan^{-1}(0)\ k \right)$$

$$= \left(\frac{1}{2}e - \frac{1}{2} \right)\hat{i} + \frac{2}{3}\ \hat{j} + \frac{\pi}{4}\ k$$

Find $\vec{r}(t)$ subject to the given conditions.

32) $\vec{r}\,'(t) = 2\ \hat{i} - 4t^3\ \hat{j} + 6\sqrt{t}\ k;\ \ r(0) = \hat{i} + 5\hat{j} + 3k$

Solution:

$$r'(t) = 2\ \hat{i} - 4t^3\ \hat{j} + 6\sqrt{t}\ k$$

$$r(t) = 2t\ \hat{i} - t^4\ \hat{j} + 4\sqrt{t^3}\ k + \vec{c}$$

Since $r(0) = \hat{i} + 5\hat{j} + 3k$

$$2(0)\ \hat{i} - (0)^4\ \hat{j} + 4\sqrt{0^3}\ k + c = \hat{i} + 5\hat{j} + 3k \Rightarrow \vec{c} = r(0) = \hat{i} + 5\hat{j} + 3k$$

Thus,

$$r(t) = 2t\ \hat{i} - t^4\ \hat{j} + 4\sqrt{t^3}\ k + \hat{i} + 5\hat{j} + 3k$$

$$r(t) = (2t+1)\ \hat{i} - \left(t^4 - 5\right)\ \hat{j} + \left(4\sqrt{t^3} + 3\right)\ k$$