MATH 267 CHAPTER 15 VECTOR-VALUED FUNCTIONS AND SPACE CURVES

15.1 VECTOR-VALUED FUNCTIONS AND SPACE CURVES

Definition Let D be a set of real numbers. A **vector-valued function** $\vec{r}$ with domain D is a correspondence that assigns to each number t in D exactly one vector $\vec{r}(t)$ in V_3.

Theorem If D is a set of real numbers, then $\vec{r}$ is a vector-valued function with domain D iff there are scalar functions *f, g,* and *h* such that

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

for every t in D.

Note We may also write $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$.

Remarks The endpoints of

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \,,$$

where the scalar functions *f, g,* and *h* are continuous on an interval **I**, determine a **space curve** (or a **curve**), i.e. a set C of all ordered triples $(f(t), g(t), h(t))$. The **graph** of C consists of all points $P(f(t), g(t), h(t))$ in an xyz – coordinate system that correspond to the ordered triples. The equations

$$x = f(t), \quad y = g(t), \quad z = h(t); \ t \text{ in } I$$

are **parametric equations** for C. We call C a **parametrized curve** and the equations a **parametrization** for C. The **orientation** of C is the direction determined by increasing values of t.

The notions of **endpoints**, **closed curve**, and **simple closed curve** are defined exactly as for plane curves (13.1). A curve C is **smooth** if it has a **smooth parametrization**:

$$x = f(t), y = g(t), z = h(t),$$

with f', g', and h' continuous and not simultaneously zero, except possibly at endpoints. A smooth curve has no sharp corners, cusps, or breaks. A **piecewise-smooth curve** is defined as in 13.1.

If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$, with *f, g,* and *h* continuous on an interval I, then as t varies through I, the endpoint of $\vec{r}(t)$ traces the curve C with parametrization

$$x = f(t), \quad y = g(t), \quad z = h(t); \ t \text{ in } I.$$

C is called **the curve determined by** $\vec{r}(t)$.

Example Describe the curve defined by $\vec{r}(t) = \langle 2+t, -1+4t, 1-t \rangle$.
Solution:
Parametric equations of the curve are

$$x = 2+t, \ \ y = -1+4t, \ \ z = 1-t$$

and we recognize these to be parametric equations of the line through $(2,-1,1)$ and parallel to the vector $\langle 1,4,-1 \rangle$.

We can write the **vector equation** of the line as $\vec{r} = \vec{r}_0 + t\vec{v}$, where $\vec{r}_0 = \langle 2,-1,1 \rangle$ and $\vec{v} = \langle 1,4,-1 \rangle$

A **twisted cubic** is a curve that has a parametrization

$$x = at, \qquad y = bt^2, \qquad z = ct^3,$$

where *a, b,* and *c* are nonzero constants.

Example 3 p. 750 Let $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ for $t \geq 0$ define a curve *C*.

Parametric equations for *C* are

$$x = t, \qquad y = t^2, \qquad z = t^3; \quad t \geq 0$$

Eliminating the parameter from the first two equations yields $y = x^2$ which implies that every point on *C* is also on the parabolic cylinder $y = x^2$.
Eliminating the parameter from the first and third yields $z = x^3$ which is an equation of a cylinder with rulings parallel to the *y* – axis. *C* is the intersection of the two cylinders.

Example 4 p. 751 Let $\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j} + bt\,\vec{k}$ for $t \geq 0$ and positive constants *a* and *b* define a curve *C*. Parametric equations are

$$x = a\cos t, \qquad y = a\sin t, \qquad z = bt; \ \ t \geq 0.$$

Eliminating the parameter in the first two yields

$$x^2 + y^2 = a^2$$

implying that every point on *C* is on the circular cylinder $x^2 + y^2 = a^2$. If *t* varies from 0 to 2π , the point *P* starts at $(a,0,0)$ and moves upward while making one revolution around the cylinder. Additional *t* – intervals of length 2π result in similar portions of the curve. *C* is called a **circular helix**.

Remark $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$ determines a curve in the *xy* – plane.

The **length of a curve** *C* in three dimensions may be defined exactly as in plane curves.

Theorem If a curve *C* has a smooth parametrization

$$x = f(t), \quad y = g(t), \quad z = h(t); \quad a \le t \le b$$

and if *C* does not intersect itself, except possibly for $t = a$ and $t = b$, then the length of *C* is

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \end{aligned}$$

Exercises

Find the length of the parametrized curve.
20) $x = t^2, \quad y = t\sin t, \quad z = t\cos t; \quad 0 \le t \le 1$
Solution:

$$x = t^2 \Rightarrow x' = 2t \qquad \Rightarrow (x')^2 = 4t^2$$

$$y = t\sin t \Rightarrow y' = \sin t + t\cos t \Rightarrow (y')^2 = \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t$$

$$z = t\cos t \Rightarrow z' = \cos t - t\sin t \Rightarrow (z')^2 = \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t$$

$$(x')^2 + (y')^2 + (z')^2 = 4t^2 + 1 + t^2 = 5t^2 + 1$$

$$L = \int_0^1 \sqrt{5t^2 + 1}\, dt \overset{Table\,\#21}{=} \ldots = \frac{1}{2}\sqrt{6} + \frac{1}{2\sqrt{5}}\ln\left(\sqrt{5} + \sqrt{6}\right) \approx 1.57$$

26) A curve *C* has the parametrization

$$x = a\sin t\sin\alpha, \quad y = b\sin t\cos\alpha, \quad z = c\cos t; \quad t \ge 0,$$

where $a, b, c,$ and α are positive constants.

a) Show that *C* lies on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

b) Show that *C* also lies on a plane that contains the *z*-axis.
c) Describe the curve *C.*

Solution:
a)

$$\begin{aligned} &\frac{(a\sin t\sin\alpha)^2}{a^2} + \frac{(b\sin t\cos\alpha)^2}{b^2} + \frac{(c\cos t)^2}{c^2} \\ &= \frac{a^2\sin^2 t\sin^2\alpha}{a^2} + \frac{b^2\sin^2 t\cos^2\alpha}{b^2} + \frac{c^2\cos^2 t}{c^2} \\ &= \sin^2 t\sin^2\alpha + \sin^2 t\cos^2\alpha + \cos^2 t \\ &= \sin^2 t\left(\sin^2\alpha + \cos^2\alpha\right) + \cos^2 t \\ &= \sin^2 t + \cos^2 t \\ &= 1 \end{aligned}$$

b) Planes that contain the *z*-axis are of the form $y = kx$, thus we show that y/x is a constant:

$$x = a\sin t\sin\alpha,\quad y = b\sin t\cos\alpha,\quad z = c\cos t;\quad t \ge 0$$

$$\frac{y}{x} = \frac{b\cancel{\sin t}\cos\alpha}{a\cancel{\sin t}\sin\alpha} = \frac{b\cos\alpha}{a\sin\alpha} = \frac{b}{a}\tan\alpha \text{ which is a constant}$$

c) *C* is the intersection of the ellipsoid in (a) and the plane in (b).

Example Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Solution:

Note that the substitutions $x = \cos t$ and $y = \sin t$ where $0 \le t < 2\pi$ will satisfy the equation of the cylinder. From the equation of the plane, we get $z = 2 - y = 2 - \sin t$. Thus, parametric equations for the curve are given by

$$x = \cos t,\quad y = \sin t,\quad z = 2 - \sin t;\quad 0 \le t < 2\pi,$$

and the vector function is given by

$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + (2 - \sin t)k = \langle \cos t,\ \sin t,\ 2 - \sin t \rangle;\ 0 \le t < 2\pi$$

Additional Exercises

1. Find a vector equation and parametric equations for the line segment that joins $P(-1,2,-2)$ to $Q(-3,5,1)$.

2. Show that the curve with parametric equations $x = t\cos t,\ \ y = t\sin t,\ \ z = t$, lies on the cone $z^2 = x^2 + y^2$, and use this to help sketch the curve.

3. Find a vector function that represents the curve of intersection of the two surfaces

a) The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$

b) The hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$.