MATH 267 CHAPTER 14 VECTORS AND SURFACES

14.5 LINES AND PLANES

Theorem: Parametric equations for the line through $P_1(x_1,y_1,z_1)$ parallel to $\vec{a}=\langle a_1,a_2,a_3\rangle$ are

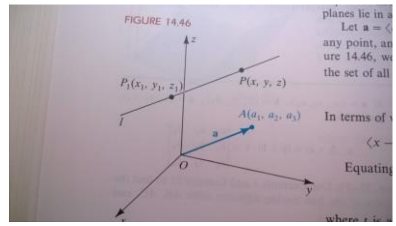
$$x = x_1 + a_1 t, \quad y = y_1 + a_2 t, \quad z = z_1 + a_3 t;\ t \text{ in } \mathbb{R}$$

Proof: Let

$\vec{a}=\langle a_1,a_2,a_3\rangle$ be a nonzero vector in V_3

$P_1(x_1,y_1,z_1)$ be any point

$\overrightarrow{OA}$ be the position vector for $\vec{a}$



The **line** l **through** $P_1(x_1,y_1,z_1)$ **parallel to** $\vec{a}$ is the set of all points $P(x,y,z)$ such that $\overrightarrow{P_1P}$ is parallel to $\overrightarrow{OA}$, i.e.

$$\overrightarrow{P_1P} = t\overrightarrow{OA} \text{ for some scalar } t$$

or,

$$\langle x-x_1, y-y_1, z-z_1\rangle = t\langle a_1,a_2,a_3\rangle,\ t\in\mathbb{R}$$

which gives

$$x = x_1 + a_1 t, \quad y = y_1 + a_2 t, \quad z = z_1 + a_3 t;\ t \text{ in } \mathbb{R}.$$

Exercises

2) Find parametric equations for the line through P parallel to a.

$$P(5,0,-2);\ \vec{a}=\langle -1,-4,1\rangle$$

Solution:

$$x = 5 - t, \quad y = -4t, \quad z = -2 + t;\ t \text{ in } \mathbb{R}$$

6) Find parametric equations for the line through $P_1(-3,1,-1)$ and $P_2(7,11,-8)$. Determine (if possible) the points at which the line intersects each of the coordinate planes.
Solution:

$a = \overrightarrow{P_1P_2} = \langle 10,10,-7\rangle$. Using $P_1(-3,1,-1)$ we get the equations

$$x = -3 + 10t, \quad y = 1 + 10t, \quad z = -1 - 7t;\ t \text{ in } \mathbb{R}$$

Line intersects the *xy*-plane when $z = 0$:

$$z = -1 - 7t = 0 \Leftrightarrow t = -1/7$$

Substitute this in *x* and *y* to get

$$x = -3 + 10(-1/7), \quad y = 1 + 10(-1/7)$$
$$x = -3 - 10/7 \qquad y = 1 - 10/7$$
$$x = -31/7 \qquad y = -3/7$$

Thus, the line intersects the *xy*-plane at $(-31/7, -3/7, -1/7)$. Similarly, the line intersects the *yz*-plane at $\left(0, \frac{34}{3}, -1\right)$ and the *xz*-plane at $\left(\frac{17}{4}, 0, \frac{13}{4}\right)$.

10) Find parametric equations for the line through the point $P(4,-1,0)$ that is parallel to the line through the points $P_1(-3,9,-2)$ and $P_2(5,7,-3)$.

Solution:

$$a = \overrightarrow{P_1P_2} = \langle 8,-2,-1\rangle$$

Thus, parametric equations of the line are

$$x = 4+8t, \quad y = -1-2t, \quad z = -t; \; t \text{ in } \mathbb{R}$$

12) Determine whether the two lines intersect, and if so, find the point of intersection.

$$l_1: \quad x = 1-6t, \quad y = 3+2t, \quad z = 1-2t$$
$$l_2: \quad x = 2+2v, \quad y = 6+v, \quad z = 2+v$$

Solution: The lines will intersect if and only if

$$\begin{aligned} 1-6t &= 2+2v \\ 3+2t &= 6+v \\ 1-2t &= 2+v \end{aligned} \quad \text{or equivalently,} \quad \begin{aligned} -6t-2v &= 1 \\ 2t-v &= 3 \\ -2t-v &= 1 \end{aligned}$$

Solving the last 2 equations, we get

$$\begin{aligned} 2t-v &= 3 \\ \underline{-2t-v} &\underline{= 1} \\ -2v &= 4 \Rightarrow v = -2 \end{aligned}$$

Substitute in the second: $2t+2 = 1 \Rightarrow 2t = 1 \Rightarrow t = 1/2$

Subsititute v and t in the first:

$$-6(1/2)-2(-2) \overset{?}{=} 1$$
$$-3+4 = 1 \text{ Yes}$$

Thus, since $v = -2$ and $t = 1/2$ satisfy the three equations, then the lines intersect and to find the point of intersection let $t = 1/2$ in the parametrization of l_1 or $v = -2$ in the parametrization of l_2 to get

$$l_1: \quad x = 1-6(1/2), \quad y = 3+2(1/2), \quad z = 1-2(1/2)$$
$$x = -2, \quad y = 4, \quad z = 0$$

OR

$$l_2: \quad x = 2+2(-2), \quad y = 6+(-2), \quad z = 2+(-2)$$
$$x = -2, \quad y = 4, \quad z = 0$$

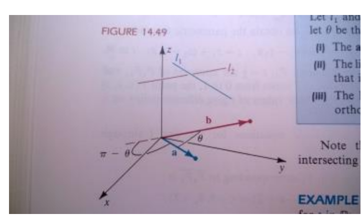
Thus, the lines intersect at $(-2,4,0)$.

Definition: Let l_1 and l_2 **be lines that are parallel to the vectors** $\vec{a}$ and $\vec{b}$, and let θ be the angle between $\vec{a}$ and $\vec{b}$.

(i) The **<u>angles between</u>** l_1 **and** l_2 are θ and $\pi-\theta$.

(ii) The lines l_1 and l_2 are **<u>parallel</u>** iff $\vec{a}$ and $\vec{b}$ are parallel, i.e., $\vec{b} = c\vec{a}$ for some scalar c.

(iii) The lines l_1 and l_2 are **<u>orthogonal</u>** iff $\vec{a}$ and $\vec{b}$ are orthogonal, i.e., $\vec{a}\cdot\vec{b} = 0$.



Exercise

16) Find the angles between l_1 and l_2 given

$$l_1: \quad x = 5+3t, \quad y = 4-t, \quad z = 3+2t$$
$$l_2: \quad x = -t, \quad y = 1-2t, \quad z = 3+t$$

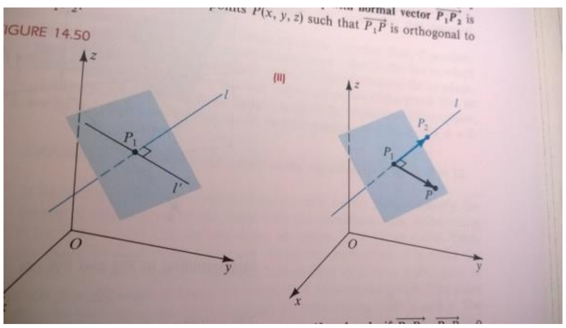
Solution:

$$\vec{a} = \langle 3,-1,2\rangle, \quad \vec{b} = \langle -1,-2,1\rangle$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{-3+2+2}{\sqrt{9+1+4}\sqrt{1+4+1}} = \frac{1}{\sqrt{14}\sqrt{6}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{14}\sqrt{6}}\right) \approx 84^\circ \text{ and } 180^\circ - \theta$$

Definition: If $P_1(x_1,y_1,z_1)$ is a point on a line l, then **<u>the plane through P_1 with normal line l</u>** is the set of points on all lines l' that are orthogonal to l at P_1. Using vectors, choose another point P_2 on l and consider the vector $\overrightarrow{P_1P_2}$. **<u>The plane through P_1 with normal vector $\overrightarrow{P_1P_2}$</u>** is the set of all points $P(x,y,z)$ such that $\overrightarrow{P_1P}$ is orthogonal to $\overrightarrow{P_1P_2}$.



Theorem: An equation of the plane through $P_1(x_1,y_1,z_1)$ with normal vector $\vec{a} = \langle a_1,a_2,a_3\rangle$ is

$$a_1(x-x_1) + a_2(y-y_1) + a_3(z-z_1) = 0.$$

Proof: A point $P(x,y,z)$ is on the plane $\Leftrightarrow \vec{a}\cdot\overrightarrow{P_1P} = 0$ or

$$\langle a_1,a_2,a_3\rangle \cdot \langle x-x_1, y-y_1, z-z_1\rangle = 0,$$

giving

$$a_1(x-x_1) + a_2(y-y_1) + a_3(z-z_1) = 0.$$

Remark: The above equation may be written in the form

$$ax + by + cz + d = 0,$$

where $a = a_1$, $b = b_1$, $c = c_1$, and $d = -a_1x_1 - a_2y_1 - a_3z_1$. …

An equation of the form $ax + by + cz + d = 0$ is called a **<u>linear equation in three variables</u>** x, y, and z.

Theorem: The graph of every linear equation $ax + by + cz + d = 0$ is a plane with normal vector $\langle a,b,c\rangle$.

Exercises Find an equation of the plane that satisfies the stated conditions.

20) Through $P(-2,5,-8)$ with normal vector

a) $\hat{i}$ **b)** $\hat{j}$ **c)** k

Solution:

a) $\hat{i} = \langle 1,0,0 \rangle$ **;** $1(x+2)+0(y-5)+0(z+8)=0 \Leftrightarrow x=-2$

b) $\hat{j} = \langle 0,1,0 \rangle$ **;** $0(x+2)+1(y-5)+0(z+8)=0 \Leftrightarrow y=5$

c) $\hat{i} = \langle 0,0,1 \rangle$ **;** $0(x+2)+0(y-5)+1(z+8)=0 \Leftrightarrow z=-8$

22) Through $P(4,2,-9)$ with normal vector $\overrightarrow{OP}$

Solution:

$$P(4,2,-9),\ \vec{a} = \langle 4,2,-9 \rangle$$
$$\pi:\quad 4(x-4)+2(y-2)-9(z+9)=0$$
$$\pi:\quad 4x+2y-9z-101=0$$

26) Through the origin and the points $P(0,2,5)$ and $Q(1,4,0)$

Solution:

$$P(0,2,5) \text{ and } Q(1,4,0)$$
$$\overrightarrow{OP} = \langle 0,2,5 \rangle \text{ and } \overrightarrow{OQ} = \langle 1,4,0 \rangle$$
$$\overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 0 & 2 & 5 \\ 1 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 4 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} k$$
$$\vec{a} = \overrightarrow{OP} \times \overrightarrow{OQ} = -20\hat{i} + 5\hat{j} - 2k$$
$$\pi:\quad -20x+5y-2z=0$$

28) Find an equation of the plane through $P(3,2,1)$, $Q(-1,1,-2)$, $R(3,-4,1)$.

Solution:

$$P(3,2,1),\ Q(-1,1,-2),\ R(3,-4,1)$$
$$\overrightarrow{PQ} = \langle -4,-1,-3 \rangle,\ \ \overrightarrow{PR} = \langle 0,-6,0 \rangle$$
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ -4 & -1 & -3 \\ 0 & -6 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -3 \\ -6 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} -4 & -3 \\ 0 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} -4 & -1 \\ 0 & -6 \end{vmatrix} k$$
$$\vec{a} = \overrightarrow{PQ} \times \overrightarrow{PR} = -18\hat{i} + 0\hat{j} + 24k$$

Using $P(3,2,1)$,

$$\pi:\quad -18(x-3)+0(y-2)+24(z-1)=0$$
$$-3(x-3)+4(z-1)=0$$
$$-3x+4z+5=0$$

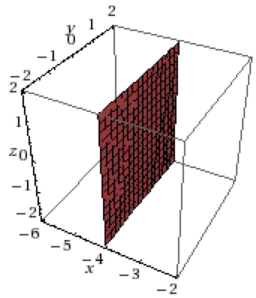
Remark: To sketch the graph of a linear equation, we find, if possible, the **<u>trace</u>** of the graph in each coordinate plane, i.e. the line in which the graph intersects the coordinate plane.

To get the trace in the *xy* – plane, set *z* = 0.
To get the trace in the *xz* – plane, set *y* = 0.
To get the trace in the *yz* – plane, set *x* = 0.

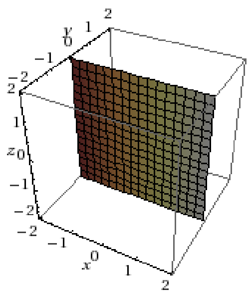
Exercise

Sketch the graph of the equation in an *xyz*-coordinate system.

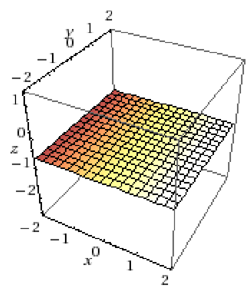
30) a) $x = -4$



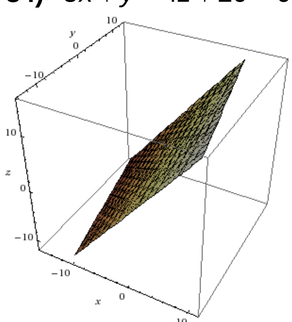
b) $y = 0$



c) $z = -\frac{2}{3}$



34) $5x + y - 4z + 20 = 0$



Find an equation of the plane.

38) See graph.
Answer: $y + z = 4$

40) See graph.
Answer: $x + 2z = 4$

Definition: Two planes with normal vectors $\vec{a}$ and $\vec{b}$ are

(i) **<u>parallel</u>** if $\vec{a}$ and $\vec{b}$ are parallel

(ii) **<u>orthogonal</u>** if $\vec{a}$ and $\vec{b}$ are orthogonal

24) Through the origin and parallel to the plane $x - 6y + 4z = 7$
Solution:

$$P(0,0,0),\ \vec{a} = \langle 1, -6, 4 \rangle$$
$$\pi:\ 1(x-0) - 6(y-0) + 4(z-0) = 0$$
$$\pi:\ x - 6y + 4z = 0$$

42) Find equation of the plane through $P(3,-2,4)$ that is parallel to the plane $-2x + 3y - z + 5 = 0$
Solution:

$$P(3,-2,4);\quad -2x + 3y - z + 5 = 0 \Rightarrow \vec{a} = \langle -2, 3, -1 \rangle$$
$$\pi:\ -2(x-3) + 3(y+2) - 1(z-4) = 0$$
$$-2x + 3y - z + 16 = 0$$

Lines may be described as intersections of planes. Given parametric equations of the line

$$x = x_1 + a_1 t, \quad y = y_1 + a_2 t, \quad z = z_1 + a_3 t;\ t \text{ in } \mathbb{R}$$

we can solve each equation for t to get

$$t = \frac{x - x_1}{a_1},\ \ t = \frac{y - y_1}{a_2},\ \ t = \frac{z - z_1}{a_3}$$

It follows that a point $P(x,y,z)$ is on l if and only if the following are satisfied:

Definition: **<u>Symmetric form</u> for a line**

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}$$

Remarks:
(1) The symmetric form is not unique since we may use any three numbers b_1, b_2, b_3 that are proportional to a_1, a_2, a_3, or any point on l other than (x_1, y_1, z_1).

(2) If we take the expressions in pairs, say

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} \text{ and } \frac{x - x_1}{a_1} = \frac{z - z_1}{a_3},$$

we obtain a description of l as an intersection of two planes, the first orthogonal to the xy – plane and the second orthogonal to the xz – plane.

(3) If one of the numbers $a_1, a_2,$ or a_3 is zero, say $a_3 = 0$, then a symmetric form is

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2},\ z = z_1,$$

which again expresses l as an intersection of two planes, the first orthogonal to the xy – plane and the second $z = z_1$ parallel to the xy – plane.

Exercise

44) Find a symmetric form for the line through $P_1(-3,1,-1)$ and $P_2(7,11,-8)$.

Solution:

$a = \overrightarrow{P_1P_2} = \langle 10,10,-7 \rangle$

Using $P_1(-3,1,-1)$, we get the symmetric equations:

$$\frac{x+3}{10} = \frac{y-1}{10} = \frac{z+1}{-7}$$

48) Find parametric equations for the line of intersection of the two planes.

$$2x + 5y + 16z = 13 \text{ and } -x - 2y - 6z = -5$$

Solution:

$$\begin{aligned} 2x + 5y + 16z = 13 &\longrightarrow \quad 2x + 5y + 16z = 13 \\ -x - 2y - 6z = -5 &\xrightarrow{\times 2} \underline{-2x - 4y - 12z = -10} \\ &\qquad\qquad y + 4z = 3 \Rightarrow y = 3 - 4z \end{aligned}$$

$$-x - 2(3 - 4z) - 6z = -5$$
$$-x - 6 + 8z - 6z = -5$$
$$x = -1 + 2z$$

Let $z = t$:

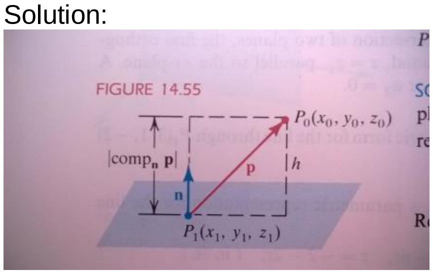
$x = -1 + 2t$

$y = 3 - 4t$

Example 13 p. 726

Find a formula for the distance h from a point $P_0(x_0, y_0, z_0)$ to the plane $ax + by + cz + d = 0$.

Solution:



Let $P_1(x_1, y_1, z_1)$ be any point in the plane and let $\vec{n}$ be a normal vector to the plane. The vector $\vec{p}$ corresponding to $\overrightarrow{P_1P_0} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle$. From the figure we see that

$$h = \left| comp_{\vec{n}} \vec{p} \right| = \left| \vec{p} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right|$$

Since $\langle a, b, c \rangle$ is a normal vector to the plane, we may let

$$\frac{\vec{n}}{\|\vec{n}\|} = \frac{\langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}}.$$

Thus,

$$h = \left| \vec{p} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right| = \left| \frac{a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$= \left| \frac{ax_0 + by_0 + cz_0 + (-ax_1 - by_1 - cz_1)}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Since P_1 is on the plane, $ax_1 + by_1 + cz_1 + d = 0$ which gives $ax_1 + by_1 + cz_1 = -d$ and thus, the

Distance from a point $P_0(x_0, y_0, z_0)$ **to the plane** $ax + by + cz + d = 0$ is

$$h = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Exercise

52) Find the distance from $P(3,1,-2)$ to the plane $2x + 4y - 5z + 1 = 0$.

Solution:

$$h = \frac{|2(3) + 4(1) - 5(-2) + 1|}{\sqrt{2^2 + 4^2 + 5^2}} = \frac{21}{\sqrt{45}} = \frac{7}{\sqrt{5}} \approx 3.13$$

54) Show that the planes $3x + 12y - 6z = -2$ and $5x + 20y - 10z = 7$ are parallel and find the distance between the planes.

Solution:

$$\vec{a} = \langle 3, 12, -6 \rangle, \quad \vec{b} = \langle 5, 20, -10 \rangle$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 3 & 12 & -6 \\ 5 & 20 & -10 \end{vmatrix} = (-120 + 120)\hat{i} - (-30 + 30)\hat{j} + (60 - 60)k = \vec{0}$$

Thus, the planes are parallel. To find the distance between the planes: Consider the point $\left(0, 0, \frac{1}{3}\right)$ on the first plane and use the formula to get

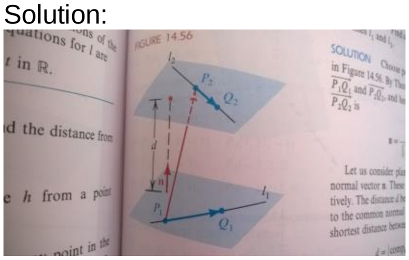
$$h = \frac{|5(0) + 20(0) - 10(1/3) - 7|}{\sqrt{5^2 + 20^2 + 10^2}} = \frac{31}{15\sqrt{21}} \approx 0.45$$

Def: Two lines are **<u>skew</u>** if they are not parallel and do not intersect.

Example 14 p. 727

Find a formula for the shortest distance *d* between two skew lines l_1 and l_2.

Solution:



Choose points P_1, Q_1 on l_1 and P_2, Q_2 on l_2. $\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}$ is orthogonal to both $\overrightarrow{P_1Q_1}$ and $\overrightarrow{P_2Q_2}$, and thus a unit vector $\vec{n}$ orthogonal to both $\overrightarrow{P_1Q_1}$ and $\overrightarrow{P_2Q_2}$ is

$$\vec{n} = \frac{\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}}{\left\| \overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2} \right\|}.$$

Consider planes through P_1 and P_2, respectively, each having normal vector $\vec{n}$. These planes are parallel and contain l_1 and l_2, respectively. The distance *d* between the planes is measured along a line parallel to the common normal $\vec{n}$. It follows that *d* is the shortest distance between l_1 and l_2. Moreover,

$$d = \left|comp_{\vec{n}} \overrightarrow{P_1P_2}\right| = \left|\vec{n} \cdot \overrightarrow{P_1P_2}\right| = \frac{\left|\left(\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\right) \cdot \overrightarrow{P_1P_2}\right|}{\left\|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\right\|}$$

Distance *d* **between two skew lines** l_1 **and** l_2 where P_1 & Q_1 are on l_1, and P_2 & Q_2 are on l_2

$$d = \frac{\left|\left(\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\right) \cdot \overrightarrow{P_1P_2}\right|}{\left\|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\right\|}$$

Exercise

56) Given: $A(1,3,0)$, $B(0,4,5)$, $C(-2,-1,2)$, $D(5,1,0)$

Let l_1 be the line through *A* and *B,* and let l_2 be the line through C and *D.* Find the shortest distance between l_1 and l_2.

Solution:

$$\overrightarrow{AB} = \langle -1,1,5 \rangle, \quad \overrightarrow{CD} = \langle 7,2,-2 \rangle, \quad \overrightarrow{AC} = \langle -3,-4,2 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ -1 & 1 & 5 \\ 7 & 2 & -2 \end{vmatrix} = (-2-10)\hat{i} - (2-35)\hat{j} + (-2-7)k$$

$$= -12\hat{i} + 33\hat{j} - 9k$$

$$\left\|\overrightarrow{AB} \times \overrightarrow{CD}\right\| = \sqrt{12^2 + 33^2 + 9^2} = \sqrt{1314}$$

$$\left(\overrightarrow{AB} \times \overrightarrow{CD}\right) \cdot \overrightarrow{AC} = 36 - 132 - 18 = -114$$

$$d = \frac{\left|\left(\overrightarrow{AB} \times \overrightarrow{CD}\right) \cdot \overrightarrow{AC}\right|}{\left\|\overrightarrow{AB} \times \overrightarrow{CD}\right\|} = \frac{114}{\sqrt{1314}} \approx 3.14$$