MATH 267 CHAPTER 14 VECTORS AND SURFACES

14.4 THE VECTOR PRODUCT

Def: A **determinant of order 2** is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

where all the letters represent real numbers.

Example: $\begin{vmatrix} -3 & 5 \\ 1 & 6 \end{vmatrix} = -18 - 5 = -23$

Def: A **determinant of order 3** is given by

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

This is also sometimes called the ***expansion of the determinant by the first row***.

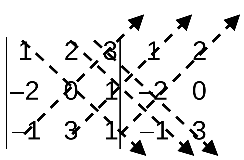
Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} -2 & 0 \\ -1 & 3 \end{vmatrix}$$

$$= 1(0-3) - 2(-2+1) + 3(-6-0)$$

$$= -3 + 2 - 18 = -19$$

Another Way of Evaluating a 3x3 Determinant



$$0 - 2 - 18 - (0 + 3 - 4) = -20 + 1 = -19$$

Definition: Given $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$. The **vector product** (or **cross product**) is

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Notation: $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

Exercise

2) Find $\vec{a}\times\vec{b}$ for $\vec{a}=\langle -5,1,-1\rangle$, $\vec{b}=\langle 3,6,-2\rangle$.

Solution:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -5 & 1 & -1\\ 3 & 6 & -2\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ -5 & 1\\ 3 & 6\end{matrix}$$

$$=-2\hat{i}-3\hat{j}-30k-\left(3k-6\hat{i}+10\hat{j}\right)$$

$$=4\hat{i}-13\hat{j}-33k$$

Theorem: The vector $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Proof: Use T14.21: Two vectors are orthogonal iff their dot product is 0. Thus, we need to show that

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{a}=0 \text{ and } \left(\vec{a}\times\vec{b}\right)\cdot\vec{b}=0$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ a_1 & a_2\\ b_1 & b_2\end{matrix}$$

$$=a_2b_3\vec{i}+a_3b_1\vec{j}+a_1b_2\vec{k}-\left(a_2b_1\vec{k}+a_3b_2\vec{i}+a_1b_3\vec{j}\right)$$

$$=\left(a_2b_3-a_3b_2\right)\vec{i}+\left(a_3b_1-a_1b_3\right)\vec{j}+\left(a_1b_2-a_2b_1\right)\vec{k}$$

Then

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{a}=\left(\left(a_2b_3-a_3b_2\right)\vec{i}+\left(a_3b_1-a_1b_3\right)\vec{j}+\left(a_1b_2-a_2b_1\right)\vec{k}\right)\cdot\left(a_1\vec{i}+a_2\vec{j}+a_3\vec{k}\right)$$

$$=\left(a_2b_3-a_3b_2\right)a_1+\left(a_3b_1-a_1b_3\right)a_2+\left(a_1b_2-a_2b_1\right)a_3$$

$$=a_1a_2b_3-a_1a_3b_2+a_2a_3b_1-a_1a_2b_3+a_1a_3b_2-a_2a_3b_1$$

$$=0$$

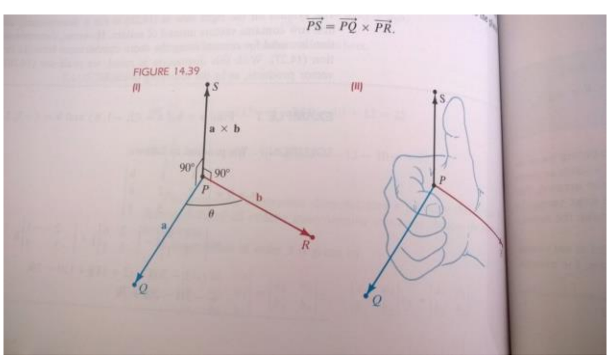
The proof for $\left(\vec{a}\times\vec{b}\right)\cdot\vec{b}=0$ is similar.

Geometric Interpretation of the Theorem

If nonzero vectors $\vec{a}$ and $\vec{b}$ correspond to vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ with the same initial point *P*, then $\vec{a}\times\vec{b}$ corresponds to a vector $\overrightarrow{PS}$ that is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Hence $\overrightarrow{PS}$ is perpendicular to the plane determined by *P*, *Q*, and *R*.

$$\overrightarrow{PS}=\overrightarrow{PQ}\times\overrightarrow{PR}.$$

The direction of $\overrightarrow{PS}$ may be obtained by using the right-hand rule.



Theorem: If θ is the angle between nonzero vectors $\vec{a}$ and $\vec{b}$, then

$$\left\|\vec{a}\times\vec{b}\right\| = \|a\|\|b\|\sin\theta .$$

Proof:

$$\left\|\vec{a}\times\vec{b}\right\|^2 = \left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{a}\times\vec{b}\right)$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$$

$$= \left(a_2b_3 - a_3b_2\right)^2 + \left(a_1b_3 - a_3b_1\right)^2 + \left(a_1b_2 - a_2b_1\right)^2$$

$$= a_2^2b_3^2 - 2a_2b_3a_3b_2 + a_3^2b_2^2 + a_1^2b_3^2 - 2a_1b_3a_3b_1 + a_3^2b_1^2 + a_1^2b_2^2 - 2a_1b_2a_2b_1 + a_2^2b_1^2$$

$$= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) - \left(a_1b_1 + a_2b_2 + a_3b_3\right)^2$$

$$\left\|\vec{a}\times\vec{b}\right\|^2 = \left(\left\|\vec{a}\right\|\left\|\vec{b}\right\|\right)^2 - \left(\vec{a}\cdot\vec{b}\right)^2$$

$$= \left(\left\|\vec{a}\right\|\left\|\vec{b}\right\|\right)^2 - \left(\left\|\vec{a}\right\|\left\|\vec{b}\right\|\right)^2\cos^2\theta$$

$$= \left(\left\|\vec{a}\right\|\left\|\vec{b}\right\|\right)^2\left(1-\cos^2\theta\right)$$

$$= \left(\left\|\vec{a}\right\|\left\|\vec{b}\right\|\right)^2\sin^2\theta$$

Take square roots and note that $\sin\theta \ge 0$ since $0 \le \theta \le \pi$, we get the desired result.

Corollary: Two vectors $\vec{a}$ and $\vec{b}$ are parallel $\Leftrightarrow$ $\vec{a}\times\vec{b} = \vec{0}$.

Proof: Trivial if either $\vec{a}$ or $\vec{b}$ is the zero vector.

Both are nonzero and are parallel $\Leftrightarrow \theta = 0$ or $\pi \Leftrightarrow \sin\theta = 0 \Leftrightarrow \vec{a}\times\vec{b} = \vec{0}$.

Exercise

12) Given: $\vec{a} = 2\hat{i} - \hat{j} + 4k,\ \vec{b} = -6\hat{i} + 3\hat{j} - 12k$

Use the vector product to show that $\vec{a}$ and $\vec{b}$ are parallel.

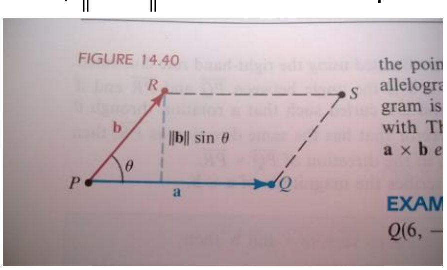
Solution:

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ 2 & -1 & 4 \\ -6 & 3 & -12 \end{vmatrix} = \begin{vmatrix} -1 & 4 \\ 3 & -12 \end{vmatrix}\hat{i} - \begin{vmatrix} 2 & 4 \\ -6 & -12 \end{vmatrix}\hat{j} + \begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix}k = (12-12)\hat{i} - (-24+24)\hat{j} + (6-6)k = \vec{0}$$

Geometric Interpretation of $\left\|\vec{a}\times\vec{b}\right\|$:

Let $\vec{a}$ and $\vec{b}$ be represented by vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ having the same initial point *P*. Let *S* be the point such that the segments *PQ* and *PR* are adjacent sides of a parallelogram with vertices *P*, *Q*, *R*, and *S*. An altitude of the parallelogram is $\left\|\vec{b}\right\|\sin\theta$, and hence its area is $\left\|\vec{a}\right\|\left\|\vec{b}\right\|\sin\theta$.

Thus, $\left\|\vec{a}\times\vec{b}\right\|$ is the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.



Exercise

16) Given: $P(-3,0,5),\ Q(2,-1,-3),\ R(4,1,-1)$

Find: **a)** a vector perpendicular to the plane determined by *P*, *Q*, and *R*.

b) the area of the triangle *PQR*

Solution:

a)

$$\overrightarrow{PQ} = \langle 5,-1,-8\rangle,\ \ \overrightarrow{PR} = \langle 7,1,-6\rangle$$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 5 & -1 & -8 \\ 7 & 1 & -6 \end{vmatrix} = \begin{vmatrix} -1 & -8 \\ 1 & -6 \end{vmatrix}\hat{i} - \begin{vmatrix} 5 & -8 \\ 7 & -6 \end{vmatrix}\hat{j} + \begin{vmatrix} 5 & -1 \\ 7 & 1 \end{vmatrix}k = 14\hat{i} - 26\hat{j} + 12k$$

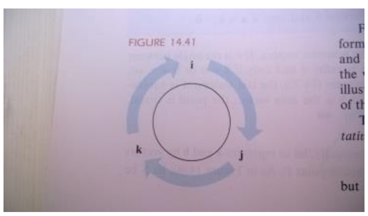
b) area = $\frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\| = \frac{1}{2}\sqrt{(14)^2+(26)^2+(12)^2} = \frac{1}{2}\sqrt{1016} = \sqrt{254}$

Properties of $\vec{i},\ \vec{j},\ \vec{k}$:

$$\vec{i}\times\vec{j} = \vec{k} \qquad \vec{j}\times\vec{k} = \vec{i} \qquad \vec{k}\times\vec{i} = \vec{j}$$

$$\vec{j}\times\vec{i} = -\vec{k} \qquad \vec{k}\times\vec{j} = -\vec{i} \qquad \vec{i}\times\vec{k} = -\vec{j}$$

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \vec{0}$$



Properties of the vector product

(i) $\vec{b}\times\vec{a} = -\left(\vec{a}\times\vec{b}\right)$

(ii) $\left(m\vec{a}\right)\times\vec{b} = m\left(\vec{a}\times\vec{b}\right) = \vec{a}\times\left(m\vec{b}\right)$

(iii) $\vec{a}\times\left(\vec{b}+\vec{c}\right) = \left(\vec{a}\times\vec{b}\right) + \left(\vec{a}\times\vec{c}\right)$

(iv) $\left(\vec{a}+\vec{b}\right)\times\vec{c} = \left(\vec{a}\times\vec{c}\right) + \left(\vec{b}\times\vec{c}\right)$

(v) $\left(\vec{a}\times\vec{b}\right)\bullet\vec{c} = \vec{a}\bullet\left(\vec{b}\times\vec{c}\right)$

(vi) $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\bullet\vec{c}\right)\vec{b} - \left(\vec{a}\bullet\vec{b}\right)\vec{c}$ **<u>triple vector product</u>**

Exercises

Prove the given property if $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle, \vec{c} = \langle c_1, c_2, c_3 \rangle$, and m is a scalar.

28) $(m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$

Proof:

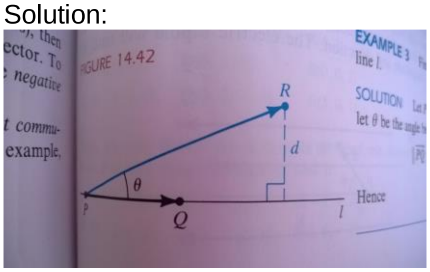
$$
\begin{aligned}
(m\vec{a}) \times \vec{b} &= \langle ma_1, ma_2, ma_3 \rangle \times \langle b_1, b_2, b_3 \rangle \\
&= \begin{vmatrix} \hat{i} & \hat{j} & k \\ ma_1 & ma_2 & ma_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} ma_2 & ma_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} ma_1 & ma_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} ma_1 & ma_2 \\ b_1 & b_2 \end{vmatrix} k \\
&= (ma_2 b_3 - ma_3 b_2)\hat{i} - (ma_1 b_3 - ma_3 b_1)\hat{j} + (ma_1 b_2 - ma_2 b_1)k \qquad (*) \\
&= m\left[(a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}\right] \\
&= m(\vec{a} \times \vec{b})
\end{aligned}
$$

Also,

$$
\begin{aligned}
(*) &= (a_2 m b_3 - a_3 m b_2)\hat{i} - (a_1 m b_3 - a_3 m b_1)\hat{j} + (a_1 m b_2 - a_2 m b_1)\hat{k} \\
&= \vec{a} \times (m\vec{b})
\end{aligned}
$$

Example 3 p. 715 Find a formula for the distance d from a point R to a line l.

Solution:



Let P and Q be points on l and let θ be the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Since $d = \|\overrightarrow{PR}\| \sin\theta$, we have

$$
\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \|\overrightarrow{PQ}\| \|\overrightarrow{PR}\| \sin\theta = \|\overrightarrow{PQ}\| d \Rightarrow d = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PQ}\|}
$$

Exercise

20) Find the distance from the point $P(-2,5,1)$ to the line through $Q(3,-1,4)$ and $R(1,6,-3)$.

Solution:

$$
\overrightarrow{QR} = \langle -2, 7, -7 \rangle, \; \overrightarrow{QP} = \langle -5, 6, -3 \rangle
$$

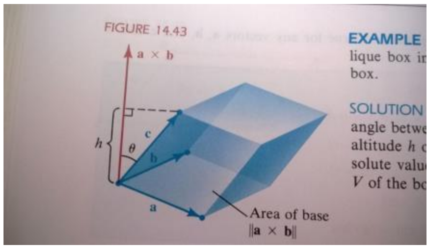
$$
\overrightarrow{QR} \times \overrightarrow{QP} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ -2 & 7 & -7 \\ -5 & 6 & -3 \end{vmatrix} = (-21 + 42)\hat{i} - (6 - 35)\hat{j} + (-12 + 35)k = 21\hat{i} + 29\hat{j} + 23k
$$

$$
\frac{\|\overrightarrow{QR} \times \overrightarrow{QP}\|}{\|\overrightarrow{QR}\|} = \frac{\sqrt{(21)^2 + (29)^2 + (23)^2}}{\sqrt{2^2 + 7^2 + 7^2}} = \frac{\sqrt{1811}}{\sqrt{102}} \approx 4.21
$$

Example 4 p. 716

Suppose $\vec{a}, \vec{b}$, and $\vec{c}$ represent adjacent sides of an oblique box. Show that $\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$ is the volume of the box.

Solution:



The area of the base of the box is $\left\|\vec{a}\times\vec{b}\right\|$. Let θ be the angle between $\vec{c}$ and $\vec{a}\times\vec{b}$. Since $\vec{a}\times\vec{b}$ is perpendicular to the base, the altitude h of the box is given by $h=\left\|\vec{c}\right\|\left|\cos\theta\right|$. Thus, the volume V of the box is

$$V=(\text{area of base})(\text{altitude})=\left\|\vec{a}\times\vec{b}\right\|\ \left\|\vec{c}\right\|\ \left|\cos\theta\right|=\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$$

We call $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$ a **<u>triple scalar product</u>**.

Exercise

21) If $\vec{a}=\langle a_1,a_2,a_3\rangle,\vec{b}=\langle b_1,b_2,b_3\rangle,\vec{c}=\langle c_1,c_2,c_3\rangle$ prove that

$$a\cdot(b\times c)=(a\times b)\cdot c=\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Proof:

$$a\cdot(b\times c)=\langle a_1,a_2,a_3\rangle\cdot\begin{vmatrix} \hat{i} & \hat{j} & k\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\langle a_1,a_2,a_3\rangle\cdot\langle b_2c_3-b_3c_2,-b_1c_3+b_3c_1,b_1c_2-b_2c_1\rangle$$

$$=(b_2c_3-b_3c_2)a_1-(b_1c_3-b_3c_1)a_2+(b_1c_2-b_2c_1)a_3 \qquad (*)$$

$$=\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Also,

$$(*)=a_1b_2c_3\underline{\underline{-a_1b_3c_2}}-a_2b_1c_3+\underline{a_2b_3c_1}+\underline{\underline{a_3b_1c_2}}\underline{-a_3b_2c_1}$$

$$=(a_2b_3-a_3b_2)c_1-(a_1b_3-a_3b_1)c_2+(a_1b_2-a_2b_1)c_3$$

$$=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$$

22) Use Example 4 and Exercise 21 to find the volume of the box having adjacent sides *AB, AC*, and *AD*. $A(0,0,0)$, $B(1,-1,2)$, $C(0,3,-1)$, $D(3,-4,1)$

Solution:

$$\overrightarrow{AB}=\langle 1,-1,2\rangle,\quad \overrightarrow{AC}=\langle 0,3,-1\rangle,\quad \overrightarrow{AD}=\langle 3,-4,1\rangle$$

$$\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\overrightarrow{AD}=\begin{vmatrix} 1 & -1 & 2\\ 0 & 3 & -1\\ 3 & -4 & 1\end{vmatrix}=1(3-4)-(-1)(0+3)+2(0-9)$$

$$=-1+3-18=-16$$

$$\text{Volume }=\left|\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\overrightarrow{AD}\right|=|-16|=16$$

24) If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are represented by vectors with a common initial point, show that

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=0 \Leftrightarrow \vec{a},\ \vec{b},\text{ and } \vec{c} \text{ are coplanar}$$

Proof: $(\Rightarrow)$

$\vec{b}\times\vec{c}$ is orthogonal to $\vec{b}$ and $\vec{c}$

$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=0\Rightarrow\vec{a}$ is also orthogonal to $\vec{b}\times\vec{c}$

Since the three vectors have a common initial point, they must all lie in the same plane.

$(\Leftarrow)$

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar, then $\vec{b}\times\vec{c}$ is orthogonal to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Thus, $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=0$.

32) Verify without using components for the vectors:

$$\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=2\left(\vec{b}\times\vec{a}\right)$$

Proof:

$$\begin{aligned}\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)&=\left(\vec{a}+\vec{b}\right)\times\vec{a}-\left(\vec{a}+\vec{b}\right)\times\vec{b}\\&=\vec{a}\times\vec{a}+\vec{b}\times\vec{a}-\vec{a}\times\vec{b}-\vec{b}\times\vec{b}\\&=\vec{0}+\vec{b}\times\vec{a}+\vec{b}\times\vec{a}+\vec{0}\\&=2\left(\vec{b}\times\vec{a}\right)\end{aligned}$$