MATH 267 CHAPTER 14 VECTORS AND SURFACES

14.3 THE DOT PRODUCT

Definition: Given $\vec{a}=\langle a_1,a_2,a_3\rangle$ and $\vec{b}=\langle b_1,b_2,b_3\rangle$, the **dot product** $\vec{a}\bullet\vec{b}$ is defined

$$\vec{a}\bullet\vec{b} = a_1b_1+a_2b_2+a_3b_3$$

The dot product is also called the **scalar product** or **inner product**.

Exercise

Given $\vec{a}=\langle -2,3,1\rangle$, $\vec{b}=\langle 7,4,5\rangle$, and $\vec{c}=\langle 1,-5,2\rangle$

Find:

2) $\vec{b}\cdot\vec{c}$

Solution:

$$\vec{b}\cdot\vec{c}=\langle 7,4,5\rangle\cdot\langle 1,-5,2\rangle = 7\cdot 1+4\cdot(-5)+5\cdot 2 = 7-20+10=-3$$

Properties of the dot product

For any vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and a scalar *c*, we have

(i) $\vec{a}\bullet\vec{a}=\|a\|^2$

(ii) $\vec{a}\bullet\vec{b}=\vec{b}\bullet\vec{a}$

(iii) $\vec{a}\bullet(\vec{b}+\vec{c})=\vec{a}\bullet\vec{b}+\vec{a}\bullet\vec{c}$

(iv) $(c\vec{a})\bullet\vec{b}=c(\vec{a}\bullet\vec{b})=\vec{a}\bullet(c\vec{b})$

(v) $\vec{0}\bullet\vec{a}=0$

Exercise

40) Prove: $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

Proof:

If $\vec{a}=\langle a_1,a_2,a_3\rangle$ and $\vec{b}=\langle b_1,b_2,b_3\rangle$ then

$$\begin{aligned}\vec{a}\cdot\vec{b} &= \langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle\\ &= a_1b_1+a_2b_2+a_3b_3\\ &= b_1a_1+b_2a_2+b_3a_3\\ &= \langle b_1,b_2,b_3\rangle\cdot\langle a_1,a_2,a_3\rangle\\ &= \vec{b}\cdot\vec{a}\end{aligned}$$

Definition: Let $\vec{a}$ and $\vec{b}$ be nonzero vectors.

(i) If $\vec{b}$ is not a scalar multiple of $\vec{a}$ and if $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the position vectors for $\vec{a}$ and $\vec{b}$, respectively, then the **angle** θ **between** $\vec{a}$ **and** $\vec{b}$ is the angle AOB of the triangle determined by points *A, O,* and *B*.

(ii) If $\vec{b}=c\vec{a}$ for some scalar *c* (i.e. $\vec{a}$ and $\vec{b}$ are parallel), then $\theta=0$ if $c>0$ and $\theta=\pi$ if $c<0$.

(iii) $\vec{a}$ and $\vec{b}$ are **orthogonal**, or **perpendicular**, if $\theta=\pi/2$.

Remark: We assume that $\vec{0}$ is parallel and orthogonal to every vector $\vec{a}$.

Theorem: If θ is the angle between nonzero vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \bullet \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta .$$

Proof: (i) If $\vec{b}$ is not a scalar multiple of $\vec{a}$, i.e. $\vec{b} \neq c\vec{a}$, then we can apply the law of cosines to triangle *AOB* (see Figure 14.33) to get

$$\|\overrightarrow{AB}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|a\|\|b\|\cos\theta$$

Thus,

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 = \left(a_1^2 + a_2^2 + a_3^2\right) + \left(b_1^2 + b_2^2 + b_3^2\right) - 2\|a\|\|b\|\cos\theta$$

which simplifies to

$$-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\|a\|\|b\|\cos\theta$$

and dividing both sides by -2 gives

$$a_1b_1 + a_2b_2 + a_3b_3 = \|a\|\|b\|\cos\theta$$

or

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta .$$

(ii) If $\vec{b}$ is a scalar multiple of $\vec{a}$, i.e. $\vec{b} = c\vec{a}$, then by properties (iv) and (v) of the dot product,

$$\vec{a} \cdot \vec{b} = \vec{a} \cdot \left(c\vec{a}\right) = c\left(\vec{a} \cdot \vec{a}\right) = c\|\vec{a}\|^2 .$$

Also,

$$\|\vec{a}\| \|\vec{b}\| \cos\theta = \|\vec{a}\| \|c\,\vec{a}\| \cos\theta = |c| \|\vec{a}\|^2 \cos\theta .$$

If $c > 0$, then $|c| = c,\ \theta = 0,$ and $|c|\|\vec{a}\|^2 \cos\theta = c\|\vec{a}\|^2$.

If $c < 0$, then $|c| = -c,\ \theta = \pi,$ and $|c|\|\vec{a}\|^2 \cos\theta = c\|\vec{a}\|^2$.

Thus, $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$.

Note: The theorem says that *the dot product of two vectors is equal to the product of their magnitudes and the cosine of the angle between the vectors.*

Corollary: If θ is the angle between nonzero vectors $\vec{a}$ and $\vec{b}$, then

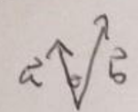
$$\cos\theta = \frac{\vec{a} \bullet \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$$

(10)

COROLLARY If θ is the angle between $\vec{a}$ & $\vec{b}$ then

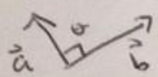
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$$

THEOREM $\vec{a}$ & $\vec{b}$ ORTHOGONAL $\Leftrightarrow$ $\vec{a}\cdot\vec{b} = 0$



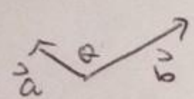
$\vec{a}\cdot\vec{b} > 0$

θ acute



$\vec{a}\cdot\vec{b} = 0$

$\theta = \pi/2$



$\vec{a}\cdot\vec{b} < 0$

θ obtuse

$\vec{a}\cdot\vec{b}$ measures the extent to which $\vec{a}$ & $\vec{b}$ point in the same direction

$\vec{a}, \vec{b}$ point in exactly the same direction

$(\theta = 0 \Rightarrow \cos\theta = 1)$

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|$

$\vec{a}, \vec{b}$ point in exactly opposite direction

$(\theta = \pi \Rightarrow \cos\theta = -1)$

$\vec{a}\cdot\vec{b} = -\|\vec{a}\|\,\|\vec{b}\|$

Exercise

Find the angle between $\vec{a}$ and $\vec{b}$.

12) $\vec{a} = \vec{i} - 7\vec{j} + 4\vec{k},\ \ \vec{b} = 5\vec{i} - \vec{k}$

Solution:

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\left\|\vec{a}\right\|\left\|\vec{b}\right\|} = \frac{\langle 1,-7,4\rangle\cdot\langle 5,0,-1\rangle}{\sqrt{1^2+7^2+4^2}\sqrt{5^2+0^2+1^2}}$$

$$= \frac{5+0-4}{\sqrt{66}\sqrt{26}} = \frac{1}{\sqrt{1716}}$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{1716}} = 88.6^\circ$$

Theorem: Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal $\Leftrightarrow$ $\vec{a}\bullet\vec{b} = 0$.

Exercise

Show that $\vec{a}$ and $\vec{b}$ are orthogonal.

16) $\vec{a} = \langle 4,-1,-2\rangle,\ \vec{b} = \langle 2,-2,5\rangle$

Solution:

$$\vec{a} = \langle 4,-1,-2\rangle,\ \vec{b} = \langle 2,-2,5\rangle$$

$$\vec{a}\cdot\vec{b} = 8+2-10 = 0$$

Find all values of c such that $\vec{a}$ and $\vec{b}$ are orthogonal.

18) $\vec{a} = 4\vec{i} + 2\vec{j} + c\vec{k},\ \ \vec{b} = \vec{i} + 22\vec{j} - 3c\vec{k}$

Solution:

$$\vec{a} = 4\vec{i} + 2\vec{j} + c\vec{k},\ \ \vec{b} = \vec{i} + 22\vec{j} - 3c\vec{k}$$

$$\vec{a}\cdot\vec{b} = 4(1) + 2(22) + c(-3c) = 4 + 44 - 3c^2 = 48 - 3c^2$$

Want $\vec{a}\cdot\vec{b} = 0$ or $48 - 3c^2 = 0$.

$$48 - 3c^2 = 0 \Leftrightarrow c^2 = 16 \Leftrightarrow c = \pm 4$$

Cauchy-Schwarz Inequality: $\left|\vec{a}\bullet\vec{b}\right| \le \left\|\vec{a}\right\|\left\|\vec{b}\right\|$

Proof:

If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, the result trivially follows.

If $\vec{a}$ and $\vec{b}$ are nonzero vectors and θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\left|\vec{a}\cdot\vec{b}\right| = \left\|\vec{a}\right\|\left\|\vec{b}\right\|\underbrace{\left|\cos\theta\right|}_{\le 1}$$

$$\le \left\|\vec{a}\right\|\left\|\vec{b}\right\|$$

Triangle Inequality: $\left\| \vec{a} + \vec{b} \right\| \le \left\| \vec{a} \right\| + \left\| \vec{b} \right\|$

Proof:

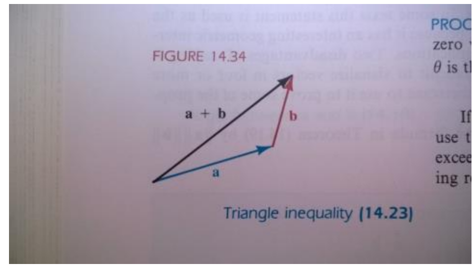


FIGURE 14.34 Triangle inequality (14.23)

$$\left\| \vec{a} + \vec{b} \right\|^2 = \left(\vec{a} + \vec{b}\right)\cdot\left(\vec{a} + \vec{b}\right)$$
$$= \vec{a}\cdot\vec{a} + 2\left(\vec{a}\cdot\vec{b}\right) + \vec{b}\cdot\vec{b}$$
$$= \left\|\vec{a}\right\|^2 + 2\left(\vec{a}\cdot\vec{b}\right) + \left\|\vec{b}\right\|^2$$

By C-S: $\vec{a}\cdot\vec{b} \le \left|\vec{a}\cdot\vec{b}\right| \le \left\|\vec{a}\right\|\left\|\vec{b}\right\|$ we have

$$\left\| \vec{a} + \vec{b} \right\|^2 \le \left\|\vec{a}\right\|^2 + 2\left\|\vec{a}\right\|\left\|\vec{b}\right\| + \left\|\vec{b}\right\|^2 = \left(\left\|\vec{a}\right\| + \left\|\vec{b}\right\|\right)^2$$

Taking square roots, we have

$$\left\| \vec{a} + \vec{b} \right\| \le \left\| \vec{a} \right\| + \left\| \vec{b} \right\|$$

Def. The angle between two directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{TR}$ is the angle between their corresponding vectors $\vec{a}$ and $\vec{b}$ in V_3.

If $\theta = \dfrac{\pi}{2}$, then $\overrightarrow{PQ}$ and $\overrightarrow{TR}$ are **orthogonal**.

Def. The **dot product** of $\overrightarrow{PQ}$ and $\overrightarrow{TR}$ is defined by

$$\overrightarrow{PQ}\cdot\overrightarrow{TR} = \vec{a}\cdot\vec{b} = \left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta = \left\|\overrightarrow{PQ}\right\|\left\|\overrightarrow{TR}\right\|\cos\theta$$

Def. If $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ have the same initial point and is S is the projection of Q on the line through P and R, then the scalar $\left\|\overrightarrow{PQ}\right\|\cos\theta$ is called the **component of** $\overrightarrow{PQ}$ along $\overrightarrow{PR}$, abbreviated

$$comp_{\overrightarrow{PR}}\overrightarrow{PQ}.$$

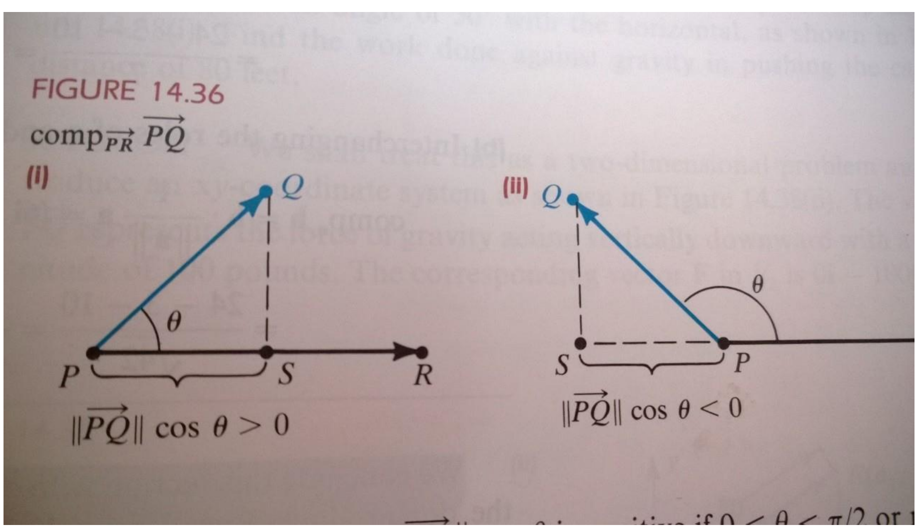


FIGURE 14.36 $\text{comp}_{\overrightarrow{PR}}\overrightarrow{PQ}$

Note: The scalar $\left\|\overrightarrow{PQ}\right\|\cos\theta$ is positive if $0 \le \theta < \dfrac{\pi}{2}$

negative if $\dfrac{\pi}{2} \le \theta < \pi$.

Thus, $comp_{\overrightarrow{PR}}\overrightarrow{PQ} = \left\|\overrightarrow{PQ}\right\|\cos\theta = \left\|\overrightarrow{PQ}\right\|\dfrac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\left\|\overrightarrow{PQ}\right\|\left\|\overrightarrow{PR}\right\|} = \dfrac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\left\|\overrightarrow{PR}\right\|}$

This proves…

Theorem The component of $\overrightarrow{PQ}$ along $\overrightarrow{PR}$ equals the dot product of $\overrightarrow{PQ}$ with the unit vector that has the same direction as $\overrightarrow{PR}$.

We use this theorem to define:

Definition: Let $\vec{a}$ and $\vec{b}$ be vectors in V_3 with $\vec{b} \neq \vec{0}$. The **<u>component of</u>** $\vec{a}$ **<u>along</u>** $\vec{b}$, is

$$\text{comp}_{\vec{b}}\vec{a} = \|\vec{a}\|\cos\theta = \frac{\vec{a}\bullet\vec{b}}{\|\vec{b}\|}.$$

Note: If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3k$ then

$\text{comp}_{\hat{i}}\vec{a} = \vec{a}\bullet\hat{i} = a_1$, $\text{comp}_{\hat{j}}\vec{a} = \vec{a}\bullet\hat{j} = a_2$, $\text{comp}_{k}\vec{a} = \vec{a}\bullet k = a_3$

i.e. the components of $\vec{a}$ along $\hat{i}, \hat{j}, k$ are the same as the components a_1, a_2, a_3

Remark The component of $\vec{a}$ along $\vec{b}$ is also called the **<u>scalar projection</u>** of $\vec{a}$ onto $\vec{b}$. We can also talk about the **<u>vector projection</u>** of $\vec{a}$ onto $\vec{b}$, denoted by $proj_{\vec{b}}\vec{a}$. This vector is defined as:

$$proj_{\vec{b}}\vec{a} = \left(comp_{\vec{b}}\vec{a}\right)\frac{\vec{b}}{\|\vec{b}\|} = \left(\frac{\vec{a}\bullet\vec{b}}{\|\vec{b}\|}\right)\frac{\vec{b}}{\|\vec{b}\|}.$$

This vector can be thought of as the "shadow" of $\vec{a}$.

Exercises

Given points $P(3,-2,-1)$, $Q(1,5,4)$, $R(2,0,-6)$, and $S(-4,1,5)$, find the indicated quantity.

20) $\overrightarrow{QS}\cdot\overrightarrow{RP}$

22) The angle between $\overrightarrow{QS}$ and $\overrightarrow{RP}$

24) The component of $\overrightarrow{QR}$ along $\overrightarrow{PS}$

Solutions:

20) $\overrightarrow{QS}\cdot\overrightarrow{RP} = \langle -5,-4,1\rangle\cdot\langle 1,-2,5\rangle = -5+8+5 = 8$

22) $\cos\theta = \dfrac{\overrightarrow{QS}\cdot\overrightarrow{RP}}{\left\|\overrightarrow{QS}\right\|\left\|\overrightarrow{RP}\right\|} = \dfrac{8}{\sqrt{5^2+4^2+1^2}\sqrt{1^2+2^2+5^2}} = \dfrac{8}{\sqrt{42}\sqrt{30}} = \dfrac{4}{\sqrt{315}} \Rightarrow \theta = \cos^{-1}\dfrac{4}{\sqrt{315}} \approx 77^\circ$

24)

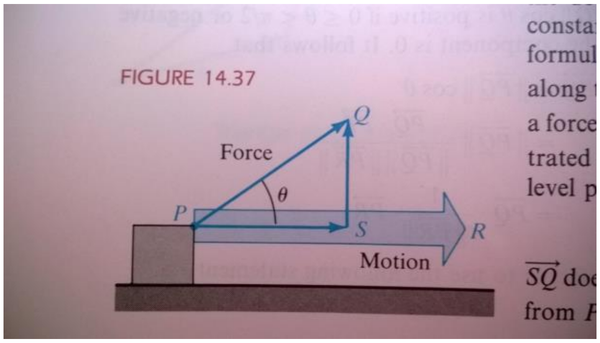
$P(3,-2,-1)$, $Q(1,5,4)$, $R(2,0,-6)$, and $S(-4,1,5)$,

$\overrightarrow{QR} = \langle 1,-5,-10\rangle$, $\overrightarrow{PS} = \langle -7,3,6\rangle$

$$\text{comp}_{\overrightarrow{PS}}\overrightarrow{QR} = \overrightarrow{QR}\bullet\frac{\overrightarrow{PS}}{\left\|\overrightarrow{PS}\right\|} = \langle 1,-5,-10\rangle\cdot\frac{\langle -7,3,6\rangle}{\sqrt{7^2+3^2+6^2}}$$

$$= \frac{-7-15-60}{\sqrt{94}} = -\frac{82}{\sqrt{94}}$$

Definition: The **work done by a constant force** $\overrightarrow{PQ}$ as its point of application moves along the vector $\overrightarrow{PR}$ is $\overrightarrow{PQ} \bullet \overrightarrow{PR}$.



$$F = \left\|\overrightarrow{PS}\right\|, d = \left\|\overrightarrow{PR}\right\|$$
$$W = Fd = \left\|\overrightarrow{PS}\right\| \left\|\overrightarrow{PR}\right\|$$
$$= \left\|\overrightarrow{PQ}\right\| \cos\theta \left\|\overrightarrow{PR}\right\| = \left\|\overrightarrow{PQ}\right\| \left\|\overrightarrow{PR}\right\| \cos\theta$$
$$= \overrightarrow{PQ} \bullet \overrightarrow{PR}$$

Exercises

26) Given: $\vec{a} = \langle 8,0,-4 \rangle$; $P(-1,2,5)$, $Q(4,1,0)$

If $\vec{a}$ represents a constant force, find the work done when its point of application moves along the line segment from P to Q.

Solution:

$$\vec{a} = \langle 8,0,-4 \rangle;\ \overrightarrow{PQ} = \langle 5,-1,-5 \rangle$$
$$\vec{a} \cdot \overrightarrow{PQ} = \langle 8,0,-4 \rangle \cdot \langle 5,-1,-5 \rangle = 40 + 0 + 20 = 60$$

28) A constant force of magnitude 5 Newtons has the same direction as the positive *z*-axis. If distance is measured in meters, find the work done if the point of application moves along a line from the origin to the point $P(1,2,3)$.

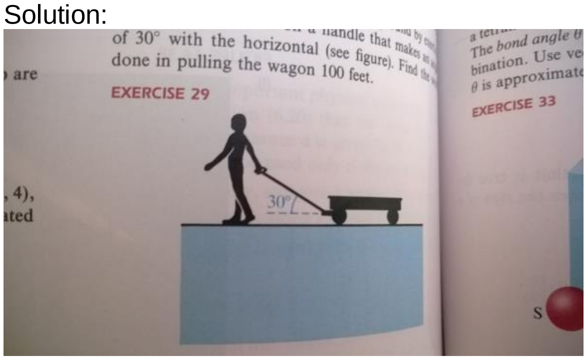
Solution:

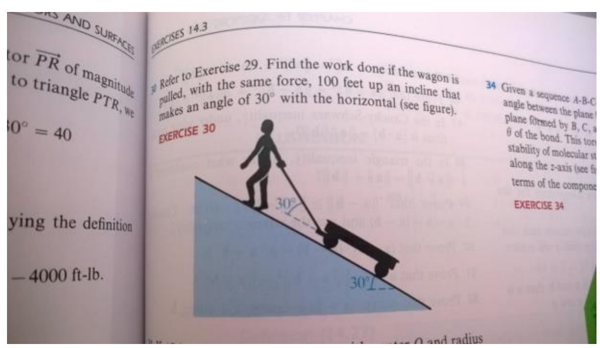
force vector $\vec{a} = \langle 0,0,5 \rangle$, direction vector $\vec{b} = \langle 1,2,3 \rangle$

$W = \vec{a} \cdot \vec{b} = 15$ joules

30) Refer to Exercise 29. Find the work done if the wagon is pulled, with the same force, 100 feet up an incline that makes an angle of 30° with the horizontal.

Solution:





force vector $\langle 20\cos 60^\circ, 20\sin 60^\circ \rangle = \langle 10, 10\sqrt{3} \rangle$

direction vector $\langle 100\cos 30^\circ, 100\sin 30^\circ \rangle = \langle 50\sqrt{3}, 50 \rangle$

work done $\langle 10, 10\sqrt{3} \rangle \cdot \langle 50\sqrt{3}, 50 \rangle = 500\sqrt{3} + 500\sqrt{3} = 1000\sqrt{3}$ ft-lb