MATH 267 CHAPTER 14 VECTORS AND SURFACES

14.2 VECTORS IN THREE DIMENSIONS

Rectangular Coordinate System in Three Dimensions

Def: An **ordered triple** (a,b,c) is a set of numbers $\{a,b,c\}$ in which *a* is considered the first number, *b* the second, and *c* the third.

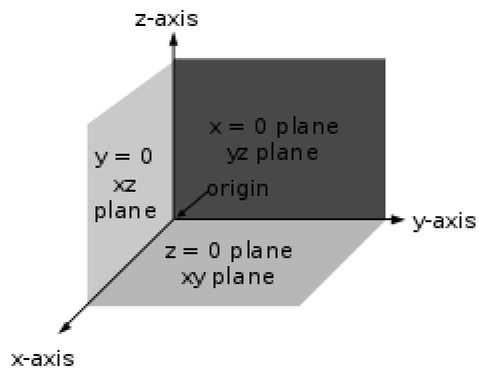
Def: Two ordered triples (a_1,a_2,a_3) and (b_1,b_2,b_3) are **equal** if

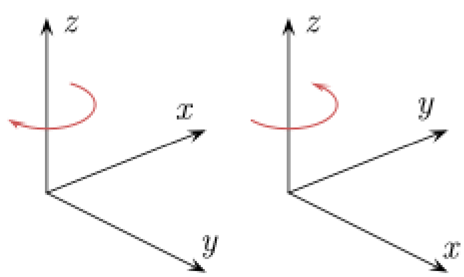
$$(a_1,a_2,a_3) = (b_1,b_2,b_3) \Leftrightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$$

Notation: $\mathbb{R}^3$

We will choose a fixed point *O* (the **origin**) and consider three mutually perpendicular **coordinate lines** (the *x-, y-,* and *z*-axes). The three coordinate lines determine three **coordinate planes**: the *xz* – plane, the *yz* – plane, and the *xy* – plane.

We will use a **right-handed** coordinate system.





The left-handed orientation is shown on the left, and the right-handed on the right.

Def: If *P* is a point, then the (perpendicular) projection of *P*

on the *x*-axis has coordinate *a*, the **x-coordinate** of *P,*
on the *y*-axis has coordinate *b*, the **y-coordinate** of *P,*
on the *z*-axis has coordinate *c*, the **z-coordinate** of *P.*

Notation: $P(a,b,c)$ or (a,b,c)

Remark: If *P* is not on a coordinate plane, then the three planes through *P* that are parallel to the coordinate planes, together with the coordinate planes, determine a rectangular box.

There is a one-to-one correspondence between the points in space and ordered triples of real numbers.

We also call this the rectangular coordinate system in three dimensions the **_xyz-coordinate system_**.

The three coordinate planes partition space into eight **octants**. The part consisting of all point $P(a,b,c)$ with *a,b,* and *c* positive is the **first octant**. The remaining octants are not numbered.

Distance between Two Points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$

(i) If P_1 and P_2 are on a line parallel to the *z*-axis, then

$$d(P_1,P_2) = |z_2 - z_1|.$$

(ii) If P_1 and P_2 are on a line parallel to the *x*-axis, then

$$d(P_1,P_2) = |x_2 - x_1|.$$

(iii) If P_1 and P_2 are on a line parallel to the *y*-axis, then

$$d(P_1,P_2) = |y_2 - y_1|.$$

(iv) If P_1 and P_2 are on a line not parallel to an axis, then

$$d(P_1,P_2) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}.$$

Vectors in Three Dimensions

We include a third component and write, for example,

$$\vec{a} = \langle a_1, a_2, a_3 \rangle,\ \vec{b} = \langle b_1, b_2, b_3 \rangle.$$

Notation: V_3 = the set of all vectors $\langle x,y,z \rangle$ where *x, y,* and *z* are real numbers

All the concepts discussed in 14.1 easily extends to V_3. For instance, the **position vector** $\overrightarrow{OA}$ is the vector with initial point at the origin *O* and terminal point at $A(a_1,a_2,a_3)$.

For $\vec{a} = \langle a_1,a_2,a_3 \rangle$, $\vec{b} = \langle b_1,b_2,b_3 \rangle$ in V_3,

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\vec{a} + \vec{b} = \langle a_1+b_1,\ a_2+b_2,\ a_3+b_3 \rangle$$

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$$

zero vector $\vec{0} = \langle 0,0,0 \rangle$

negative of $\vec{a} = -\vec{a} = \langle -a_1, -a_2, -a_3 \rangle$

unit vectors $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$

$$\vec{a} = \langle a_1,a_2,a_3 \rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

Midpoint Formula

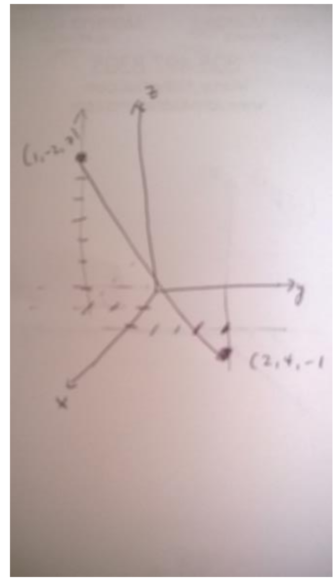
The **midpoint** of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

Exercise

2) Plot $A(1,-2,7)$ and $B(2,4,-1)$ and find

- **a)** $d(A,B)$
- **b)** midpoint of AB
- **c)** the vector in V_3 that corresponds to $\overrightarrow{AB}$.



Solution:

a) $d(A,B) = \sqrt{(1-2)^2 + (-2-4)^2 + (7-(-1))^2} = \sqrt{1+36+64} = \sqrt{101}$

b) midpoint $M = \left(\frac{1+2}{2}, \frac{-2+4}{2}, \frac{7+(-1)}{2}\right) = \left(\frac{3}{2}, 1, 3\right)$

c) $\overrightarrow{AB} = \langle 2-1, 4-(-2), -1-7\rangle = \langle 1, 6, -8\rangle$

8) Given: $\vec{a} = \langle 1,2,-3\rangle$, $\vec{b} = \langle -4,0,1\rangle$. Find

- **a)** $\vec{a}+\vec{b} = \langle 1-4, 2+0, -3+1\rangle = \langle -3,2,-2\rangle$
- **b)** $\vec{a}-\vec{b} = \langle 1-(-4), 2-0, -3-1\rangle = \langle 5,2,-4\rangle$
- **c)** $5\vec{a}-4\vec{b} = 5\langle 1,2,-3\rangle - 4\langle -4,0,1\rangle = \langle 5,10,-15\rangle + \langle 16,0,-4\rangle = \langle 21,10,-19\rangle$
- **d)** $\|\vec{a}\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$
- **e)** $\|-3\vec{a}\| = \|\langle -3,-6,9\rangle\| = \sqrt{9+36+81} = \sqrt{9(1+4+9)} = 3\sqrt{14}$

18) Given: $\vec{a} = \langle -6,-3,6\rangle$ Find the vector that has

- **a)** the same direction as $\vec{a}$ and twice the magnitude of $\vec{a}$
  Solution:
  $2\vec{a} = 2\langle -6,-3,6\rangle = \langle -12,-6,12\rangle$
- **b)** the opposite direction of $\vec{a}$ and one-third the magnitude of $\vec{a}$
  Solution:
  $-\frac{1}{3}\vec{a} = -\frac{1}{3}\langle -6,-3,6\rangle = \langle 2,1,-2\rangle$
- **c)** the same direction as $\vec{a}$ and magnitude 2
  Solution:
  $2\frac{\vec{a}}{\|\vec{a}\|} = 2\frac{\langle -6,-3,6\rangle}{\sqrt{36+9+36}} = \frac{\langle -12,-6,12\rangle}{9} = \frac{1}{3}\langle -4,-2,4\rangle$ or $\frac{2}{3}\langle -2,-1,2\rangle$ or $\left\langle -\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right\rangle$

Def: The **graph of an equation** in three variables is the set of all points $P(a,b,c)$ such that the ordered triple is a solution of the equation. The graph is called a **surface**.

An equation of a **sphere** with radius *r* and center $P_0(x_0, y_0, z_0)$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2.$$

Exercises

24) Find an equation of the sphere with center $C(3,-1,2)$ that is tangent to the

- **a)** the *xy*-plane
- **b)** the *xz*-plane
- **c)** the *yz*-plane

28) Find an equation of the sphere that has center $(2,3,-1)$ and contains the point $(1,7,-9)$.
Solution:
Find the radius by finding the distance between the two given points.
$(2,3,-1),(1,7,-9)$

$d=\sqrt{(2-1)^2+(3-7)^2+(-1-(-9))^2}=\sqrt{1+16+64}=\sqrt{81}=9$ radius

Equation is $(x-2)^2+(y-3)^2+(z+1)^2=81$.

32) Find the center and radius of the sphere $4x^2+4y^2+4z^2-4x+8y-3=0$.
Solution:

$$4x^2+4y^2+4z^2-4x+8y-3=0$$
$$4x^2-4x+4y^2+8y+4z^2-3=0$$
$$4\left(x^2-x+\frac{1}{4}\right)+4\left(y^2+2y+1\right)+4z^2=3+1+4$$
$$\left(x-\frac{1}{2}\right)^2+(y+1)^2+z^2=2 \quad \text{center } \left(\frac{1}{2},-1,0\right); \text{ radius } \sqrt{2}$$