MATH 267 CHAPTER 14 VECTORS AND SURFACES

14.1 VECTORS IN TWO DIMENSIONS

Preliminaries

scalar quantity – quantity that can be completely characterized by a single real number
examples: area, volume, time, arc length, temperature

vector – characterized by both magnitude and direction
– represented by a **directed line segment**
ex. $\overrightarrow{PQ}$ with **initial point** P and **terminal point** Q
$\left\| \overrightarrow{PQ} \right\|$ = the **magnitude** of $\overrightarrow{PQ}$ and is the length of the line segment PQ

equivalent vectors – vectors with the same magnitude and direction
– regarded as **equal** in calculus

Examples from physics:
- velocity vector
- force vector

displacement – term used to represent the path of a point or particle as it moves along a line segment from A to B
$= \overrightarrow{AB}$

sum of 2 displacements or 2 vectors as a particle moves from A to B and then from B to C

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

$\overrightarrow{AC}$ is called the **resultant force**, i.e., the single force that produces the same effect as the two combined forces.

If c is a scalar and $\vec{v}$ is a vector, then $c\vec{v}$ is called a **scalar multiple** of $\vec{v}$.

$$\left\| c\vec{v} \right\| = c\left\| \vec{v} \right\|$$

$c > 0 \Rightarrow c\vec{v}$ is in the same direction as $\vec{v}$

$c < 0 \Rightarrow c\vec{v}$ is in the opposite direction of $\vec{v}$

Vectors in the Cartesian Plane

position vector – vector with initial point at the origin
$\overrightarrow{OA}$ vector with initial point at $(0,0)$ and terminal point (a_1, a_2)

There is a one-to-one correspondence between vectors in an xy – plane and ordered pairs of real numbers.

Notation: $\vec{a} = \langle a_1, a_2 \rangle$, where a_1 and a_2 are called the **components** of $\vec{a}$

V_2 = the set of all vectors $\langle x, y \rangle$ where x and y are real numbers

Definitions: Given $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$ vectors in V_2

$$\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1 \text{ and } a_2 = b_2$$

$$\left\| \vec{a} \right\| = \left\| \langle a_1, a_2 \rangle \right\| = \sqrt{a_1^2 + a_2^2}$$

$$\vec{a} + \vec{b} = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right)$$

$$\vec{0} = \langle 0, 0 \rangle$$

$$-\vec{a} = -\langle a_1, a_2 \rangle = \langle -a_1, -a_2 \rangle$$

$$c\vec{a} = c\langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle$$

Properties of Addition and Scalar Multiples of Vectors

(i) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(ii) $\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$

(iii) $\vec{a} + \vec{0} = \vec{a}$

(iv) $\vec{a} + \left(-\vec{a}\right) = \vec{0}$

(v) $c\left(\vec{a} + \vec{b}\right) = c\vec{a} + c\vec{b}$

(vi) $(c+d)\vec{a} = c\vec{a} + d\vec{a}$

(vii) $(cd)\vec{a} = c\left(d\vec{a}\right) = d\left(c\vec{a}\right)$

(viii) $1\vec{a} = \vec{a}$

(ix) $0\vec{a} = \vec{0} = c\vec{0}$

Theorem: If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any points, the vector $\vec{a}$ in V_2 that **corresponds to** $\overrightarrow{P_1P_2}$ is

$$\vec{a} = \langle x_2 - x_1, y_2 - y_1 \rangle .$$

Remark: If $\vec{a}$ and $\vec{b}$ are arbitrary vectors, then

$$\vec{b} + \left(\vec{a} - \vec{b}\right) = \vec{a} ;$$

i.e. $\vec{a} - \vec{b}$ is the vector that, when added to $\vec{b}$, gives $\vec{a}$.

Definition: Nonzero vectors $\vec{a}$ and $\vec{b}$ in V_2

(i) have the **<u>same direction</u>** if $\vec{b} = c\vec{a}$ for some scalar $c > 0$

(ii) have the **<u>opposite direction</u>** if $\vec{b} = c\vec{a}$ for some scalar $c < 0$

(iii) are **<u>parallel</u>** if $\vec{b} = c\vec{a}$ for some scalar *c*, i.e., if they are (i) or (ii)

Theorem: $\left\| c\vec{a} \right\| = |c| \left\| \vec{a} \right\|$

Definition: $\vec{i} = \langle 1,0 \rangle$, $\vec{j} = \langle 0,1 \rangle$

$\left\| \vec{i} \right\| = 1$, $\left\| \vec{j} \right\| = 1$

Theorem: If $\vec{a} = \langle a_1, a_2 \rangle$, then $\vec{a} = a_1\vec{i} + a_2\vec{j}$ ($\vec{i},\vec{j}$ form)

Remarks:

(i) In the $\vec{i},\vec{j}$ form, a_1 is called the **<u>horizontal component</u>** and a_2 is called the **<u>vertical component</u>** of vector $\vec{a}$.

(ii) The vector sum $a_1\vec{i} + a_2\vec{j}$ is called a **<u>linear combination</u>** of $\vec{i}$ and $\vec{j}$.

If $\vec{a} = a_1\vec{i} + a_2\vec{j}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j}$ and *c* is a scalar, then

$$\left(a_1\vec{i} + a_2\vec{j}\right) + \left(b_1\vec{i} + b_2\vec{j}\right) = (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j}$$
$$\left(a_1\vec{i} + a_2\vec{j}\right) - \left(b_1\vec{i} + b_2\vec{j}\right) = (a_1 - b_1)\vec{i} + (a_2 - b_2)\vec{j}$$
$$c\left(a_1\vec{i} + a_2\vec{j}\right) = (ca_1)\vec{i} + (ca_2)\vec{j}$$

Definition: A **<u>unit vector</u>** is a vector of magnitude 1.

Theorem: If $\vec{a} \neq \vec{0}$, then the unit vector $\vec{u}$ that has the same direction as $\vec{a}$ is

$$\vec{u} = \frac{\vec{a}}{\left\| \vec{a} \right\|}.$$

Exercises

Use components to express the sum or difference as a scalar multiple of one of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$, or $\vec{f}$.

$\vec{a} = \langle 2,0 \rangle$, $\vec{b} = \langle -1,0 \rangle$, $\vec{c} = \langle 0,2 \rangle$, $\vec{d} = \langle 0,-1 \rangle$, $\vec{e} = \langle 2,2 \rangle$, $\vec{f} = \langle 1,2 \rangle$

14) $\vec{c} - \vec{d}$

Solution: $\vec{c} - \vec{d} = \langle 0,2 \rangle - \langle 0,-1 \rangle = \langle 0-0, 2-(-1) \rangle = \langle 0,3 \rangle = -3\vec{d}$

16) $\vec{f} - \vec{b}$

Solution: $\vec{f} - \vec{b} = \langle 1,2 \rangle - \langle -1,0 \rangle = \langle 1-(-1), 2-0 \rangle = \langle 2,2 \rangle = \vec{e}$

18) $\vec{e} + \vec{c}$

Solution: $\vec{e} + \vec{c} = \langle 2,2 \rangle + \langle 0,2 \rangle = \langle 2,4 \rangle = 2\vec{f}$

20) Given: $P(7,-3)$ and $Q(-2,4)$ Find the vector $\vec{a}$ in V_2 that corresponds to $\overrightarrow{PQ}$. Sketch $\overrightarrow{PQ}$ and the position vector for $\vec{a}$.

Solution: $\overrightarrow{PQ} = \langle -2-7, 4-(-3) \rangle = \langle -9,7 \rangle$

32) Find a vector of magnitude 4 that has the opposite direction of $\vec{a} = \langle 2, -5 \rangle$.
Solution:

$$\vec{a} = \langle 2, -5 \rangle \Rightarrow \|\vec{a}\| = \sqrt{2^2 + 5^2} = \sqrt{29}$$

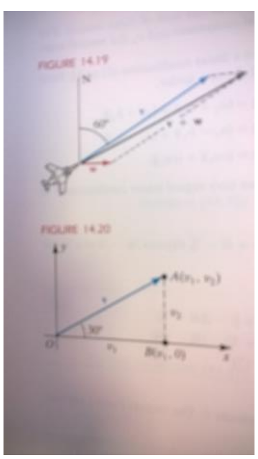
$$\frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\sqrt{29}}\langle 2, -5 \rangle \text{ or } \left\langle \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}} \right\rangle$$ unit vector in the same direction as $\vec{a}$

$$-4\frac{\vec{a}}{\|\vec{a}\|} = \frac{-4}{\sqrt{29}}\langle 2, -5 \rangle \text{ or } \left\langle \frac{-8}{\sqrt{29}}, \frac{20}{\sqrt{29}} \right\rangle$$ vector of magnitude 4 and in opposite direction as $\vec{a}$

For the following, approximate the horizontal and vertical components of the vector that is described.

36) A child pulls a sled through the snow by exerting a force of 20 pounds at an angle of 40° with the horizontal.
Solution: Horizontal component is $20\cos 40^0 =$ and the vertical component is $20\sin 40^0 =$

Example 7 p. 692
In air navigation, directions are specified by measuring from the north in a *clockwise* direction. Suppose an airplane with an airspeed of 200 mph is flying in the direction 60°, and a wind is blowing directly from the west at 40 mph. These velocities may be represented by vectors $\vec{v}$ and $\vec{w}$ of magnitudes 200 and 40, respectively. The direction of the resultant $\vec{v} + \vec{w}$ is the **true course** of the airplane relative to the ground, and the magnitude $\|\vec{v} + \vec{w}\|$ is the **ground speed** of the airplane. Approximate the ground speed to the nearest mph and the true course to the nearest degree.



Solution:
Consider an *xy*-plane with the airplane at the origin, the *y*-axis on the north-south line, and $\vec{v} = \langle v_1, v_2 \rangle$. Since $\|\vec{v}\| = 200$, it follows that

$$v_1 = 200\cos 30^\circ = 100\sqrt{3} \quad \text{and} \quad v_2 = 200\sin 30^\circ = 100$$

Thus,

$$\vec{v} = \langle 100\sqrt{3}, 100 \rangle \text{ and } \vec{w} = \langle 40, 0 \rangle$$

The resultant force is

$$\vec{v} + \vec{w} = \langle 100\sqrt{3} + 40, 100 \rangle,$$

and the ground speed is

$$\|\vec{v} + \vec{w}\| = \sqrt{(100\sqrt{3} + 40)^2 + (100)^2} \approx 235 \text{ mph}.$$

If θ is the angle between $\vec{v} + \vec{w}$ and the positive *x*-axis, then

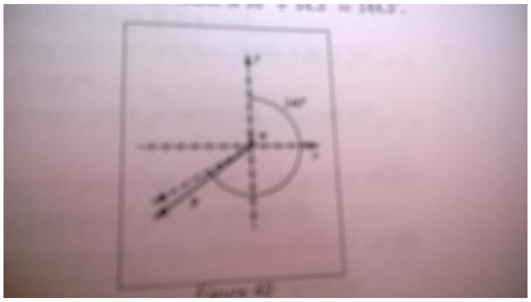
$$\tan\theta = \frac{100}{100\sqrt{3} + 40} \approx 0.469 \text{ and } \theta \approx \tan^{-1}(0.469) \approx 25^\circ$$

Hence the true course is approximately $90^\circ - 25^\circ = 65^\circ$.

Exercise

40) An airplane pilot wishes to maintain a true course in the direction 240° with a ground speed of 400 mph when the wind is blowing directly north at 50 mph. Find the required airspeed and compass heading.

Solution: We are given the resultant vector (true course) $\vec{v}+\vec{w}$ and the wind vector $\vec{w}$. We need to find $\vec{v}$.



Now,

$$\vec{v}+\vec{w} = 400\left\langle -\cos 30^0, -\sin 30^0 \right\rangle = 400\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle = \left\langle -200\sqrt{3}, -200 \right\rangle$$

$$\vec{w} = 50\langle 0,1\rangle = \langle 0,50\rangle$$

$$\vec{v} = \left(\vec{v}+\vec{w}\right) - \vec{w} = \left\langle -200\sqrt{3}, -200 \right\rangle - \langle 0,50\rangle = \left\langle -200\sqrt{3}, -250 \right\rangle$$

$$\left\|\vec{v}\right\| = \sqrt{\left(-200\sqrt{3}\right)^2 + \left(-250\right)^2} = \sqrt{182{,}500} \approx 427.2 \text{ mph}$$

$$\tan^{-1}\left(\frac{-250}{-200\sqrt{3}}\right) \approx 35.8^0 \Rightarrow \text{direction is } 270^0 - 35.8^0 = 234.2^0$$