1 A Review of Arithmetic

Before we begin our study of algebra, let us briefly go over important concepts and methods we learned in arithmetic. The same concepts and methods can be used in algebra. We will first introduce some notations, then review what real numbers are and the properties governing the operations on real numbers, then go over the sign rules when performing operations, and go over rules governing operations on fractions, mixed numbers, and decimals.

1.1 Sets and Notations

A set is a collection of objects, numbers, letters, etc. A member of a set is called an element of the set. Uppercase letters are usually used to denote sets. The symbol \{ \} is used to enclose the elements of a set. The symbol \(\in\) is used to denote that something is an element of a set; otherwise, the symbol \(\notin\) is used.

Example 1. The following show the notation on the left and the meaning on the right.

\[
\begin{align*}
N &= \{1, 2, 5\} & N \text{ is the set whose elements are } 1, 2, \text{ and } 5. \\
A &= \{w, x, y, z\} & A \text{ is the set whose elements are } w, x, y, \text{ and } z. \\
P &= \{\text{cat, dog, pig}\} & P \text{ is the set whose elements are cat, dog, and pig.} \\
2 \in N & \quad 2 \text{ is an element of } N. \\
bird \notin P & \quad \text{bird is not an element of } P.
\end{align*}
\]

When the elements of a set \(A\) are contained entirely in another set \(B\), we say that \(A\) is a subset of \(B\) and we write \(A \subseteq B\).

Example 2. If \(A = \{1, 2, 3\}\) and \(B = \{1, 2, 3, 4, 5, 6\}\), then \(A \subseteq B\) and \(A\) is called a proper subset of \(B\) since it does not contain all the elements of \(B\). The empty set or null set, denoted by \(\{\}\) or \(\emptyset\) is also a subset of \(B\) and it is called an improper subset of \(B\). Another improper subset of \(B\) is the set \(B\) itself. If we try to generate all the subsets of \(B\), we should produce \(2^6\) or 64 subsets. In general, a set with \(n\) elements has \(2^n\) subsets.
Example 3 Find all the subsets of \( P = \{x, y, z\} \).

Solution: The empty set \( \emptyset \) and \( P \) itself are subsets of \( P \). Next, the sets \( \{x\}, \{y\}, \) and \( \{z\} \) are subsets of \( P \). Then the sets \( \{x,y\}, \{x,z\}, \) and \( \{y,z\} \).

We have exhausted all the possibilities and produced 8 or \( 2^3 \) subsets of \( P \).

If we can list all the elements of a set, we call that set a **finite set**. If it is impossible to list all the elements of a set, then that set is called an **infinite set**. We will see several examples of infinite sets in the next section. For example, the set of counting numbers \( \mathbb{N} \) is infinite and we can write \( \mathbb{N} = \{1, 2, 3, \ldots\} \), with the three dots at the end denoting that the list goes on forever.

<table>
<thead>
<tr>
<th>Other Notations</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Rightarrow )</td>
<td>“implies”; used in place of “If…then…”</td>
</tr>
<tr>
<td>( \Leftrightarrow )</td>
<td>“if and only if”</td>
</tr>
<tr>
<td>( \ni ) or ( \varnothing )</td>
<td>“such that”</td>
</tr>
<tr>
<td>( \exists )</td>
<td>“there exists”</td>
</tr>
<tr>
<td>( \forall )</td>
<td>“for every” or “for all”</td>
</tr>
<tr>
<td>( \wedge )</td>
<td>“and”</td>
</tr>
<tr>
<td>( \lor )</td>
<td>“or”</td>
</tr>
<tr>
<td>( \therefore )</td>
<td>“therefore” or “thus”</td>
</tr>
<tr>
<td>( \because )</td>
<td>“because”</td>
</tr>
</tbody>
</table>

1.2 The Real Numbers

Let us now look at what the real numbers are. What are numbers? What are “real” numbers?

The word **number** is used to denote the concept or idea of measure: length, width, height, how many days, how much money, how much liquid, etc. The visual representation of a number is a **numeral**. There are several numeration systems, but the most widely-used is the **Hindu-Arabic decimal system** which uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Computer systems use the **binary system** which uses only **two** digits: 0 and 1. The **Roman numeral system** uses the symbols I, V, X, L, C, D, and M whose values are 1, 5, 10, 50, 100, 500, and 1000, respectively.
Let us now develop the real number system using the Hindu-Arabic numerals that we know.

We start off with the numbers that we learned when we were toddlers – we counted 1, 2, 3, etc. These numbers are called the counting numbers or the natural numbers. Mathematicians denote this set of numbers by the Greek letter $\mathbb{N}$. Thus, we write $\mathbb{N} = \{1, 2, 3, \ldots\}$. Natural numbers are also referred to as positive integers.

Then we learned the notion of eating up all our candy and having nothing left – so we need a number to represent nothing and mathematicians decided to use 0 and call the set consisting of 0 and the natural numbers the set of whole numbers and denote this set by the Greek letter $\mathbb{W}$. Thus, $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$.

As we got older, we experienced not having enough money to buy things, so we borrowed money from a friend or a bank – we need to denote this money that we borrowed and thus were born the negative numbers. For now, let us talk of the negative integers – these are simply the (positive) integers preceded by the negative sign. So, we can think of $-20$, for example, as having borrowed $20$ from a friend. This new larger set is called the set of integers and is denoted by $\mathbb{Z}$, where $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Gordon, I will gladly pay you Tuesday for $10$ today.

I have $20$… $-10$
Then we learned about parts of a whole when we cut up a candy bar, a pie, pizza, etc., and we need to represent these parts of a whole, so the set of fractions or mixed numbers had to be created. For example, the fraction \( \frac{2}{3} \) can represent the fact that we ate 2 candy bars (numerator is 2) from a bag containing 3 candy bars (denominator is 3), or the mixed number \( 1\frac{3}{4} \) to represent the fact that almost 2 whole pizzas were eaten.

Fractions and mixed numbers belong to the set of rational numbers because these numbers can be written as the ratio of two integers, where the denominator cannot be 0. This set is denoted by \( \mathbb{Q} \) and is defined by

\[
\mathbb{Q} = \{\frac{a}{b} \mid a \text{ and } b \text{ are integers, } b \neq 0\}.
\]

Note that unlike the previous sets where we enumerated the elements of the set, we need to describe the elements of \( \mathbb{Q} \) since it is impossible to enumerate all of them. The first part of the description is what the elements look like: \( \frac{a}{b} \) - a ratio; the second part after the | (read as “such that”) tells us what \( a \) and \( b \) are.

This way of describing a set is called the rule method because the second part gives the rule or property that the elements of the set must satisfy. The other way of defining a set is called the roster method – this is when we list all the elements of the set. For example, \( A = \{-1, 0, 1\} \) is the set whose elements are \(-1, 0, \) and \(-1\).
**Fractions** are obviously rational. The numerator can be 0 since, for example, \(0/21=0\) and 0 is a real number. The denominator **cannot** be 0 since, for example, \(21/0=\text{error}\) if you use a calculator. This is the same notion as dividing by 0: we **never** divide by 0. We will write **UND** for undefined when we get such an expression.

**Mixed numbers** are rational because we can write them as fractions. For example, \(1\frac{3}{4}=7/4\). We will review in Section 1.3 how to convert a mixed number to a fraction.

**Integers** are also rational since we can write any integer as the integer on the numerator and 1 on the denominator. For example, \(21=21/1\).

**Decimal numbers** or **decimals** are used to denote parts of a whole that was divided into 10 or 100 or 1000, etc. pieces. We use decimals when we talk of money (for example $12.99 is the cost of a shirt) or distance (0.1 mile - we say a tenth of a mile), etc. These decimals are also rational because \(12.99=1299/100\) and \(0.1=1/10\). We will review decimals in Section 1.4.

Decimals like 12.99 and 0.1 are called **terminating decimals**. Another class of decimals that can be written as fractions consist of decimals that do not terminate or end but **repeat**. For example, it can be shown that 0.33333… can be written as \(1/3\). Thus, **non-terminating** but **repeating decimals** are rational.

Decimals that do not terminate nor repeat belong to the final set of real numbers: the **irrational numbers**. Examples of irrational numbers are \(\sqrt{2} \approx 1.41421356237\) and \(\pi \approx 3.14159265359\). These numbers are real numbers because we can “see” them: \(\sqrt{2}\) is the length of the hypotenuse of a right triangle whose legs are length 1 each; \(\pi\) is the ratio of the circumference of a circle and its diameter. The set of irrational numbers is denoted by \(\mathbb{Q}'\) - symbol \(...'\) (read “prime”) means the **complement** of the set of rational numbers.
The following summarizes the preceding discussion.

Real Numbers $\mathbb{R}$

- Rational Numbers $\mathbb{Q}$
  - Fractions (or decimals that terminate or repeat)

- Irrational Numbers $\mathbb{Q}'$
  - Nonterminating and nonrepeating decimals

Integers $\mathbb{Z}$

Whole Numbers $\mathbb{W}$

Natural Numbers $\mathbb{N}$

We can also picture the real numbers using a **Venn Diagram**.

The Venn diagram illustrates the notion of subset explained in the previous section. We see that

$$N \subset W \subset Z \subset Q \subset \mathbb{R},$$

since the sets keep getting “larger”. Likewise, $Q' \subset \mathbb{R}$. Finally, the union of $Q$ and $Q'$ is $\mathbb{R}$ and we write $\mathbb{R} = Q \cup Q'$.

Another way to have a picture of the real numbers is by using a **number line**. Each point on the number line corresponds to a real number called the **coordinate** of the point, and conversely, each real number corresponds to a point on the number line. The real number 0 is assigned to a point on the
number line. All the points to the left of 0 correspond to the negative real numbers; all the points to the right of 0 correspond to the positive real numbers.

negative real numbers (-)  positive real numbers (+)

In symbols, for a real number $a$, we write (and read as)

$a > 0$ (a is **greater than** 0) to denote that $a$ is positive;

$a < 0$ (a is **less than** 0) to denote that $a$ is negative

To find the points corresponding to the fractions $1/3$ and $-3/4$:

4 pieces 3 pieces

To find the point corresponding to the fraction $16/5$, divide 16 by 5 to get the mixed number $3\frac{1}{5}$:

5 pieces

To find the point corresponding to the decimal number -2.8:

10 pieces (recall that .8=8/10)
To find the point 1.16, divide the interval between 1 and 2 into 100 pieces and mark the point 16 units to the right of 1.

Finally, let us define the notion of the absolute value of a real number. In terms of the number line, the absolute value of a real number is the distance of the point corresponding to the real number from the point corresponding to the real number 0. We use the absolute value bars $|$ to denote this concept. For example, $|4| = 4$, $|-12.1| = 12.1$, and of course $|0| = 0$. We note that the absolute value of a real number is either 0 or positive, it is never negative.

1.3 Properties of Real Numbers

In the following, we will use the letters $a$, $b$, and $c$ to denote real numbers and the symbol $\in$ to denote that a real number belongs to some set. The properties will be stated for the operations of addition and multiplication – we all know what addition means and we know that multiplication is repeated addition.
1.3.1 The Closure Properties

Addition: If \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \), then \( a + b \in \mathbb{R} \).

Multiplication: If \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \), then \( ab \in \mathbb{R} \).

The sum or product of two real numbers is a real number.

1.3.2 The Commutative Properties

Addition: If \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \), then \( a + b = b + a \).

Multiplication: If \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \), then \( ab = ba \).

The order in which we add or multiply two real numbers is not important.

1.3.3 The Associative Properties

Addition: If \( a, b, c \in \mathbb{R} \), then \( a + (b + c) = (a + b) + c \).

Multiplication: If \( a, b, c \in \mathbb{R} \), then \( (ab)c = a(bc) \).

Grouping the addends or factors is not important.

We don’t need to put grouping symbols when we’re simply adding or multiplying real numbers.

1.3.4 The Distributive Properties of Multiplication over Addition

Left Distributive Property: If \( a, b, c \in \mathbb{R} \), then \( a(b + c) = ab + ac \).

Right Distributive Property: If \( a, b, c \in \mathbb{R} \), then \( (b + c)a = ba + ca \).

We can either perform the addition first before multiplying the factor \( a \), or we can multiply \( a \) to each of the addends before performing the addition.
1.3.5 The Existence of Identity Elements

**Addition:** If \( a \in \mathbb{R} \), then there is \( 0 \in \mathbb{R} \) such that \( a + 0 = 0 + a = a \).

**Multiplication:** If \( a \in \mathbb{R} \), then there is \( 1 \in \mathbb{R} \) such that \( a \cdot 1 = 1 \cdot a = a \).

The **identity element for addition** is \( 0 \) since any number added to \( 0 \) gives the number; the **identity element for multiplication** is \( 1 \) since any number multiplied to \( 1 \) gives the number.

1.3.6 The Existence of Inverse Elements

**Addition:** If \( a \in \mathbb{R} \), then there is \( -a \in \mathbb{R} \) such that
\[
a + (-a) = -a + a = 0.
\]

**Multiplication:** If \( a \in \mathbb{R} \) and \( a \neq 0 \), then there is \( 1/a \in \mathbb{R} \) such that
\[
a \cdot (1/a) = (1/a) \cdot a = 1.
\]

The **additive inverse** of a real number is the **negative** or **opposite** of the real number. The **multiplicative inverse** of a **nonzero** real number is the **reciprocal** of the real number.

The notion of the additive inverse gives us another way to define the **absolute value** of a real number \( a \):
\[
|a| = \begin{cases} a, & \text{if } a \text{ is nonnegative} \\ -a, & \text{if } a \text{ is negative} \end{cases}
\]
Using more symbols, we can write the above definition as:

\[
|a| = \begin{cases} 
a, & a \geq 0 \\
-a, & a < 0
\end{cases}
\]

This is consistent with what we observed that the absolute value of a real number is never negative. For example, if \(a\) is \(-5\), then

\[
|a| = |-5| = -\left(-\frac{5}{a}\right) = 5.
\]

**Example 1.** State the property that justifies the given statement.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. (2 + (-5) = -5 + 2)</td>
<td>commutative property for addition</td>
</tr>
<tr>
<td>b. (12 + 0 = 12)</td>
<td>identity element for addition is 0</td>
</tr>
<tr>
<td>c. (-12 + 12 = 0)</td>
<td>additive inverse property</td>
</tr>
<tr>
<td>d. ((-3+1) + 5 = -3 + (1+5))</td>
<td>associative property for addition</td>
</tr>
<tr>
<td>e. (-\frac{2}{5} \cdot 1 = -\frac{2}{5})</td>
<td>identity element for multiplication is 1</td>
</tr>
<tr>
<td>f. (-\frac{2}{5} \cdot \left(-\frac{5}{2}\right) = 1)</td>
<td>multiplicative inverse property</td>
</tr>
<tr>
<td>g. (7(5 + 8) = 7 \cdot 5 + 7 \cdot 8)</td>
<td>distributive property of multiplication over addition</td>
</tr>
</tbody>
</table>

**Example 2.** Use the distributive property of multiplication over addition to remove the parentheses. Assume the letters represent real numbers.

a. \(3(a + b)\)  
b. \(4(x + y + z)\)

**Solution:**

a. \(3(a + b) = 3 \cdot a + 3 \cdot b \) or \(3a + 3b\)

b. \(4(x + y + z) = 4 \cdot x + 4 \cdot y + 4 \cdot z = 4x + 4y + 4z\)
1.4 Operations on Real Numbers

We shall assume that we all know what addition means. Multiplication is repeated multiplication; exponentiation (raising a real number to a nonnegative integer) is repeated multiplication.

With the notion of inverses, we can also now define the operations of subtraction and division.

Let $a$ and $b$ be real numbers. We define subtraction of $a$ and $b$ as

$$a - b = a + (-b)$$

Thus, the difference of $a$ and $b$ is the sum of $a$ and the opposite or additive inverse of $b$.

Let $a$ and $b$ be real numbers, with $b$ nonzero. We define division as

$$a \div b = \frac{a}{b}$$

Thus, the quotient of $a$ and $b$, with $b$ nonzero, is the product of $a$ and the reciprocal or multiplicative inverse of $b$. We can also write $a \div b$ as $\frac{a}{b}$. 
Sign Rules

Let us now review how to perform the operations of addition, subtraction, multiplication, division, and exponentiation on signed numbers.

1.4.1 Addition

The sum of real numbers with the same sign is a real number with the same sign as the addends. This means that when we add positive real numbers, the sum is positive; when we add negative real numbers, the sum is negative.

Example 1.

a. \(2 + 5 + 6 + 9 = 22\)  
b. \(-2 + (-5) + (-6) + (-9) = -22\)

When we add two real numbers with opposite signs, what we do is we subtract the real number with the smaller absolute value from the real number with the larger absolute value and the resulting sum will have the sign of the real number with the larger absolute value.

Example 2.

a. \(12 + (-7) = 5\)  
b. \(-12 + 7 = -5\)

Subtract 7 from 12;  
The 1st answer is positive because the “larger” number is positive…the 2nd answer is negative because the “larger” is negative.

When adding several real numbers with different signs, we can either work from left to write performing the addition on two numbers at a time, or, we can add all the positive numbers together and all the negative numbers together first to get the final sum.

Example 3.  \(1 + 2 + (-3) + 4 + (-5)\)

Solution: Either go from left to right:

\[
1 + 2 + (-3) + 4 + (-5) = 3 + (-3) + 4 + (-5) = 0 + 4 + (-5) = 4 + (-5) = -1
\]

Or, add the positives and negatives first:

\[
1 + 2 + (-3) + 4 + (-5) = 1 + 2 + 4 + (-3) + (-5) = 7 + (-8) = -1
\]
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1.4.2 Subtraction

We use the definition of subtraction $a - b = a + (-b)$ and then apply the rules governing addition. The first number $a$ is called the **minuend** and the second number $b$ is called the **subtrahend**.

**Example 4.**

a. $4 - 9 = 4 + (-9) = -5$  
   b. $-4 - 9 = -4 + (-9) = -13$  
   c. $4 - (-9) = 4 + 9 = 13$  
   d. $-4 - (-9) = -4 + 9 = 5$

1.4.3 Multiplication

The sign of the **product** $ab$ will depend on the sign of each of the **factors** $a$ and $b$.

If the factors $a$ and $b$ are both positive, the product $ab$ is **positive**.
If the factors $a$ and $b$ are both negative, the product $ab$ is **positive**.
If one factor is **positive** and the other factor is **negative**, the product $ab$ is **negative**.

**Example 5.**

a. $5 \cdot 4 = 20$  
   b. $5 \cdot (-4) = -20$  
   c. $(-5) \cdot 4 = -20$  
   d. $(-5) \cdot (-4) = 20$

When there are **three or more factors**, the product will be **positive** if there are an even number of negative factors, and **negative** if there are an odd number of negative factors.

**Example 6.**

a. $(-1) \cdot 2 \cdot (-3) \cdot 4 = 24$  
   b. $(-1) \cdot (-2) \cdot (-3) \cdot 4 = -24$
1.4.4 Division

We use the definition of division \( a \div b = a \cdot \frac{1}{b} \) or \( a = a \cdot \frac{1}{b} \) where \( b \neq 0 \) and follow the rules governing multiplication of two real numbers. This means that when both the dividend \( a \) and the divisor \( b \) have the same sign, then the quotient \( \frac{a}{b} \) is positive. When \( a \) and \( b \) have opposite signs, then \( \frac{a}{b} \) is negative.

Example 7.
\[
\begin{align*}
\text{a. } & \quad 24 \div 4 = 6 \\
\text{b. } & \quad -24 \div 4 = -6 \\
\text{c. } & \quad -24 \div (-4) = 6 \\
\text{d. } & \quad 24 \div (-4) = -6
\end{align*}
\]

or equivalently,
\[
\begin{align*}
\text{a. } & \quad \frac{24}{4} = 6 \\
\text{b. } & \quad \frac{-24}{4} = -6 \\
\text{c. } & \quad \frac{-24}{-4} = 6 \\
\text{d. } & \quad \frac{24}{-4} = -6
\end{align*}
\]

1.4.5 Exponentiation

For any nonzero real number \( b \) and a positive integer \( n \), we define the power \( b^n \) as

\[
b^n = b \cdot b \cdot \ldots \cdot b
\]

We call \( b \) the base and \( n \) the exponent of the exponential expression. We see that the rules governing multiplication can be applied. When the base \( b \) is positive, \( b^n \) is positive for any positive integer \( n \).

The more interesting case is when the base \( b \) is negative,

\( b^n \) is positive if \( n \) is a positive even integer;
\( b^n \) is negative if \( n \) is a positive odd integer.

Example 8.  
\[
\begin{align*}
\text{a. } & \quad (-1)^{12} = 1 \\
\text{b. } & \quad (-1)^{33} = -1 \\
\text{c. } & \quad (-2)^2 = 4 \\
\text{d. } & \quad (-2)^3 = -8
\end{align*}
\]
1.4.6 Combined Operations

When two or more operations appear together in one problem, it is important to know which operation should be performed first. The following gives guidelines on how we should proceed:

1. Perform all the operations **inside symbols of grouping** first.
2. Perform all the **exponentiations** next.
3. Perform all the **multiplications and divisions** in the order in which they appear from **left to right**.
4. Perform all the **additions and subtractions** last.

We will use the following abbreviations to justify the steps in the examples:
- **P** for parentheses
- **M** for multiplication
- **A** for addition
- **E** for exponentiation
- **D** for division
- **S** for subtraction

**Example 9.** Evaluate: \(-4 + 2(9 - 6) - 3 \cdot 2^2 ÷ 6\)

Solution:
\[
(1) P \text{ and } E \quad 6 \\
(2) M \ 
(3) D \ 
(4) A/S \\
\]

**Example 10.** Evaluate: \((2 - 5)^2 + 2^2 - 5^2\)

Solution:
\[
P \text{ and } E \ 
E \text{ and } S \\
S = 12 \\
\]

**Example 11.** Evaluate: \(-3^2 - (-2)^2 + (-1)^4 \div (5 - 3)^2 + 4 ÷ 2\)

Solution:
\[
E, P, D \ 
S, E \ 
A \ 
D = 2 \\
\]
1.5 Fractions, Mixed Numbers, Decimals and Percent

Let us now review how to perform operations on non-integers without using a calculator.

1.5.1 Simplifying Fractions

Let us first review how to simplify a fraction or reduce a fraction to lowest terms. To simplify a fraction \( \frac{a}{b} \) \((b \neq 0)\), we divide out the common factors from the numerator \( a \) and the denominator \( b \).

Example 1.

a. \( \frac{12}{20} = \frac{\cancel{2} \cdot 6}{\cancel{2} \cdot 10} = \frac{\cancel{2} \cdot 3}{\cancel{2} \cdot 5} = \frac{3}{5} \)

b. \( -\frac{24}{15} = -\frac{\cancel{3} \cdot 8}{\cancel{3} \cdot 5} = -\frac{8}{5} \)

In (b), we could have divided 4 by 2 first:

\( \frac{4}{9} = \frac{\cancel{2} \cdot 5}{9 \cdot 2} = \frac{\cancel{2} \cdot 5}{9} = \frac{10}{9} \)

"cancel" means divide!!

1.5.2 Multiplying Fractions

To multiply fractions, simply multiply across, that is, multiply numerator to numerator and denominator to denominator.

Example 2.

a. \( \frac{1}{2} \cdot \frac{3}{4} = \frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8} \)

b. \( \frac{4}{9} \cdot \frac{5}{2} = \frac{20}{18} = \frac{\cancel{2} \cdot 10}{\cancel{2} \cdot 9} = \frac{10}{9} \)

In (b), we could have divided 4 by 2 first:

\( \frac{4}{9} = \frac{\cancel{2} \cdot 5}{9 \cdot 2} = \frac{\cancel{2} \cdot 5}{9} = \frac{10}{9} \)

Just do it…just multiply…top with top, bottom with bottom….

Make sure the final answer is in lowest terms.
1.5.3 Dividing Fractions

To divide a fraction by a second fraction, multiply the first fraction by the reciprocal of the second fraction.

Example 3.

\[
a. \quad \frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \cdot \frac{3}{2} = \frac{12}{10} = \frac{6}{5}
\]

\[
b. \quad \frac{12}{13} \div 36 = \frac{12}{13} \cdot \frac{1}{36} = \frac{12}{13} \cdot \frac{1}{36} = \frac{1}{39}
\]

1.5.4 Adding or Subtracting Fractions

The only fractions that can be added or subtracted right away are those with the same denominator.

Example 4.

\[
a. \quad \frac{2}{5} + \frac{1}{5} = \frac{3}{5}
\]

\[
\text{c.} \quad \frac{2}{5} + \frac{1}{5} = \frac{1}{5}
\]

\[
b. \quad \frac{2}{5} - \frac{1}{5} = \frac{1}{5}
\]

\[
d. \quad \frac{2}{5} - \frac{1}{5} = \frac{3}{5}
\]

When the denominators are not the same, we first need to find a common denominator, preferably the lowest common denominator (LCD). This is the (lowest) common multiple (LCM) of the denominators, that is, the (lowest) number that each denominator can divide evenly.

Example 5.

The LCM of 2 and 3 is 6; other common multiples are 12, 18, etc.

The LCM of 4 and 8 is 8; other common multiples are 16, 24, etc.

After determining the LCD, the next thing to do is to find the equivalent fraction that has the LCD for its denominator. For example, to change \(\frac{2}{3}\) to a fraction with denominator 6, we need to do the following:

\[
\frac{2}{3} = \frac{2 \cdot 2}{3 \cdot 2} = \frac{4}{6}.
\]
Example 6. Find: \( \frac{1}{2} + \frac{2}{3} \)

Solution: The LCM of 2 and 3 is 6 so the LCD is 6. Next, we need to change each fraction to a fraction with denominator 6. To do this, we need to multiply 1/2 by 3/3 and 2/3 by 2/2:

\[
\frac{1}{2} + \frac{2}{3} = \frac{1}{2} \cdot \frac{3}{3} + \frac{2}{3} \cdot \frac{2}{2} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6}
\]

Example 7. Find: \( 1 + \frac{3}{5} - \frac{3}{10} \)

Solution: The LCD is 10. We need to change each fraction to an equivalent fraction with denominator 10:

\[
1 + \frac{3}{5} - \frac{3}{10} = \frac{10}{10} + \frac{3}{5} \cdot \frac{2}{2} - \frac{3}{10} = \frac{10 + 6}{10} - \frac{3}{10} = \frac{13}{10}
\]

When the denominators share some common factors, an efficient way to find the LCD is to use prime factorization and lining up the factors that are the same.

Example 8. Find the LCD: \( \frac{5}{6}, \frac{1}{8}, \) and \( \frac{7}{12} \)

Solution:

\[
\begin{align*}
6 &= 2 \cdot 3 \\
8 &= 2 \cdot 2 \cdot 2 \\
12 &= 2 \cdot 2 \cdot 3 \\
\text{LCD} &= 2 \cdot 2 \cdot 3 = 24
\end{align*}
\]

This method gives us the lowest common denominator.

Thus, the lowest common denominator or LCD is the product of the greatest power of each factor occurring in a factorization. In the example, the LCD can also be written as \( 2^3 \cdot 3 \).
1.5.5 Operations on Mixed Numbers

An easy way to perform operations on mixed numbers is to change the mixed numbers to fractions and follow the rules governing fractions.

To change a mixed number to a pure fraction, we multiply the denominator to the whole number and add the numerator to get the numerator of the pure fraction.

Example 9. Change the given mixed number to a fraction.

a. $4\frac{2}{3}$

Solution: $a. \quad 4\frac{2}{3} = 4 + \frac{2}{3} = \frac{12}{3} + \frac{2}{3} = \frac{14}{3}$

b. $-2\frac{5}{7}$

Solution: $b. \quad -2\frac{5}{7} = -2 + \frac{5}{7} = \frac{-14}{7} + \frac{5}{7} = \frac{-19}{7}$

Example 10. Perform the given operation.

a. $4\frac{2}{3} \cdot (-2\frac{5}{7})$

Solution: $a. \quad 4\frac{2}{3} \cdot (-2\frac{5}{7}) = \frac{14}{3} \cdot \frac{-19}{7} = \frac{-28}{3} \cdot \frac{19}{7} = \frac{-38}{3}$

b. $4\frac{2}{3} \div (-2\frac{5}{7})$

Solution: $b. \quad 4\frac{2}{3} \div (-2\frac{5}{7}) = \frac{14}{3} \div \frac{-19}{7} = \frac{14}{3} \cdot \frac{7}{-19} = \frac{-98}{57}$

c. $4\frac{2}{3} - 2\frac{5}{7}$

Solution: $c. \quad 4\frac{2}{3} - 2\frac{5}{7} = \frac{14}{3} - \frac{19}{7} = \frac{14}{3} \cdot \frac{7}{7} - \frac{19}{7} \cdot \frac{3}{3} = \frac{98}{21} - \frac{57}{21} = \frac{41}{21}$

Note that we just left our answers as fractions – this is perfectly okay. If we want to change them back to mixed numbers, we simply divide the numerator by the denominator to get $-12\frac{2}{3}$, $-1\frac{41}{57}$, and $1\frac{20}{21}$ for (a), (b), and (c), respectively.
1.5.6 Operations on Decimals

Let us review how to perform operations on decimals. When a decimal point is not shown, it is understood to be immediately to the right of the last digit of a number. For example:

\[21 = 21.0 \quad -4 = 4.0\]

To add or subtract decimals, simply align the decimal points and add/subtract as usual.

**Example 11.** Add: \(0.2 + 0.001 + 21.01\)

Solution: We do the following (we may add the trailing zeros if we want to):

\[
\begin{array}{c}
0.200 \\
0.001 \\
21.010 \\
\hline
21.211 \\
\end{array}
\]

To multiply decimals, we can first ignore the decimal points and multiply as if the numbers are integers, and then put the decimal point to the answer and the position of the decimal point will correspond to the total number of decimal places counting from the right.

**Example 12.** Multiply: \(1.1 \times 0.01 \times 10.001\)

Solution: We multiply \(11 \times 1 \times 10001\) to get 110011. Now, there are \(1+2+3 = 6\) total decimal places from each of the factors, so the answer is 0.110011 (count 6 places from the right of the answer obtained by removing the decimal points).

To divide a decimal number by another decimal number, multiply both numbers by an appropriate power of 10 to make the divisor an integer.

**Example 13.** Divide: \(3.21 \div 0.2\)

Solution: We multiply both numbers by 10 (this will make 0.2 the integer 2) and divide \(32.1 \div 2\).
1.5.7 Fraction, Decimal, Percent

To change a fraction to a decimal, simply divide the numerator by the denominator.

**Example 14.** Convert the given fraction to a decimal.

a. \( \frac{3}{4} \)  
Solution: \( \frac{3}{4} = 3 \div 4 = 0.75 \)

b. \( \frac{-1}{3} \)
Solution: \( \frac{-1}{3} = -1 \div 3 = -0.333... = -\frac{1}{3} \)

To change a terminating decimal to a fraction, write the decimal as a fraction with an appropriate power of 10 for its denominator. Reduce to lowest terms.

**Example 15.** Convert the given terminating decimal to a fraction.

a. 0.2  
b. −1.01  
c. 11.111

Solution:

a. \( 0.2 = \frac{2}{10} = \frac{2 \cdot 1}{5} = \frac{1}{5} \)

b. \( -1.01 = -\frac{101}{100} \)

c. \( 11.111 = \frac{11111}{1000} \)

To change a percent to a decimal, replace the % symbol by 1/100 and write the resulting fraction as a decimal. Recall that dividing by 100 is the same as moving the decimal point 2 places to the left.

I think we can just use a calculator!?

It’s good to know the methods…?
Example 16. Convert the given percent to a decimal.

a. 2%  
Solution:
\[ 2\% = 2 \cdot \frac{1}{100} = \frac{2}{100} = 0.02 \]

b. 0.04%  
Solution:
\[ 0.04\% = 0.04 \cdot \frac{1}{100} = \frac{0.04}{100} = 0.0004 \]

To change a decimal to a percent, multiply the decimal number by 100%. Recall that multiplying by 100 is the same as moving the decimal point 2 places to the right.

Example 17. Convert the given decimal to a percent.

a. 0.7  
Solution:
\[ 0.7 \cdot 100\% = 70\% \]

b. 3.2  
Solution:
\[ 3.2 \cdot 100\% = 320\% \]

1.5.8 Introduction to Roots of Real Numbers

We will learn later that the roots of a real number correspond to raising the real number to a rational exponent. That is, finding the root is the same as performing exponentiation where the exponent is a fraction.

A positive real number has two square roots, one positive, the other negative. We define a square root of a positive real number \( a \) as a real number \( b \) such that \( b^2 = a \).

Example 18. Give the two square roots of the given real number.

a. 4  
Solution:
\[ \text{The two square roots of 4 are } 2 \text{ and } -2, \text{ since } 2^2 = 4 \text{ and } (-2)^2 = 4. \]

b. 25  
Solution:
\[ \text{The two square roots of 25 are } 5 \text{ and } -5, \text{ since } 5^2 = 25 \text{ and } (-5)^2 = 25. \]

c. \( \frac{9}{16} \)  
Solution:
\[ \text{The two square roots of } \frac{9}{16} \text{ are } \frac{3}{4} \text{ and } -\frac{3}{4}, \text{ since } \left(\frac{3}{4}\right)^2 = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16} \text{ and } \left(-\frac{3}{4}\right)^2 = \left(-\frac{3}{4}\right) \cdot \left(-\frac{3}{4}\right) = \frac{9}{16}. \]
d. The two square roots of 0.04 are 0.2 and −0.2, since \((0.2)^2 = 0.04\) and \((-0.2)^2 = 0.04\).

The positive square root is called the **principal square root**. We write \(\sqrt{a}\) (read as “square root of \(a\)”) to denote the principal square root.

**Example 19.** Evaluate the following.

a. \(\sqrt{4}\) \hspace{2cm} b. \(\sqrt{25}\) \hspace{2cm} c. \(\sqrt[2]{\frac{9}{16}}\) \hspace{2cm} d. \(\sqrt{0.04}\)

Solution:

a. \(\sqrt{4} = 2\) \hspace{2cm} b. \(\sqrt{25} = 5\) \hspace{2cm} c. \(\sqrt[2]{\frac{9}{16}} = \frac{3}{4}\) \hspace{2cm} d. \(\sqrt{0.04} = 0.2\)

A nonzero real number has **three cube roots**, **four fourth roots**, etc. A real number has only **one** principal cube root, fourth root, etc. We will discuss these in Chapter 6.