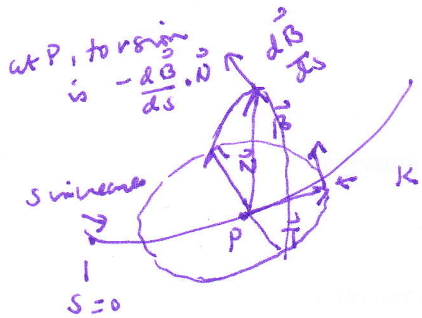
CURVATURE, TORSION, TNB FRAME



$\kappa = \left|\frac{d\vec{T}}{ds}\right|$ at P

$\vec{B} = \vec{T} \times \vec{N}$

binormal vector

curvature $\kappa = \left|\frac{d\vec{T}}{ds}\right|$

– how much a vehicle's path turns to the left or right as it moves along

torsion $-\frac{d\vec{B}}{ds}\cdot\vec{N}$

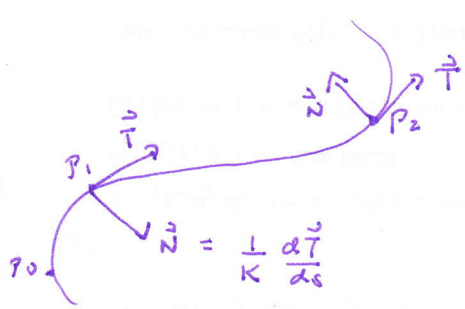
– how much a vehicle's path rotates or twists out of its plane of motion as the vehicle moves along

EVERY MOVING BODY TRAVELS WITH A <u>TNB FRAME</u> THAT CHARACTERIZES THE GEOMETRY OF ITS PATH OF MOTION.

ex: P = train climbing up a curved track

curvature = rate at which the headlight turns from side to side per unit distance

torsion = rate at which the engine tends to twist out of the plane formed by $\vec{T}$ & $\vec{N}$



$\vec{T}$ turns clockwise

$\vec{N}$ points toward the right

$\vec{T}$ turns counterclockwise

$\vec{N}$ points toward the left

$\therefore$ $\vec{N}$ will point toward the concave side of the curve

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CIRCLE & RADIUS OF CURVATURE

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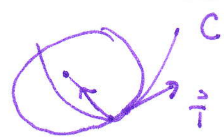
OSCULATING CIRCLE



$r = \frac{1}{\kappa}$

FOR $\kappa \neq 0$, the circle of curvature is the circle in the plane of the curve that

(1) is tangent to the curve at P (has the same tangent line the curve has)

(2) has the same curvature the curve has at P

(3) lies toward the concave or inner side of the curve

CURVATURE FOR SPACE CURVES

Ex. $\vec{r}(t) = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j} + bt\,\hat{k}, \quad a, b \geq 0, \quad a^2+b^2 \neq 0$

$$\vec{v}(t) = \vec{r}\,'(t) = -(a\sin t)\,\hat{i} + (a\cos t)\,\hat{j} + b\,\hat{k}$$

$$\|\vec{v}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$$

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{-(a\sin t)\,\hat{i} + (a\cos t)\,\hat{j} + b\,\hat{k}}{\sqrt{a^2+b^2}}$$

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}$$

$$= \frac{d\vec{T}}{dt}\cdot\frac{1}{\|\vec{v}\|} \qquad \text{since } \frac{ds}{dt} = \|\vec{v}\|$$

$$= \frac{-(a\cos t)\,\hat{i} - (a\sin t)\,\hat{j}}{\sqrt{a^2+b^2}}\cdot\frac{1}{\sqrt{a^2+b^2}}$$

$$= \frac{-(a\cos t)\,\hat{i} - (a\sin t)\,\hat{j}}{a^2+b^2} = \frac{a}{a^2+b^2}\left[-(\cos t)\,\hat{i} - (\sin t)\,\hat{j}\right]$$

$$K = \left|\frac{d\vec{T}}{ds}\right| = \frac{a}{a^2+b^2}\left\|-\cancel{a}(\cos t)\,\hat{i} - (\sin t)\,\hat{j}\right\|$$

$$= \frac{a}{a^2+b^2}\sqrt{\cos^2 t + \sin^2 t} = \frac{a}{a^2+b^2}$$

NOTE: 1) $\vec{r}(t)$ describes a helix

2) Increasing b for a fixed a decreases the curvature.

3) Decreasing a for a fixed b eventually decreases the curvature as well.

Stretching a spring tends to straighten it.

4) If $b = 0$, the helix reduces to a circle of radius a & $K = \frac{1}{a}$.

5) If $a = 0$, the helix becomes the z-axis & $K = 0$.