18.6. The Divergence Theorem

<u>Divergence Theorem</u> Let Q be a region in 3D bounded by a closed surface S, and let $\vec{n}$ denote the unit outer normal to S at (x,y,z). If $\vec{F}$ is a vector function that has continuous partial derivatives on Q, then

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_Q \nabla\cdot\vec{F}\,dV;$$

i.e., the flux of $\vec{F}$ over S equals the triple integral of the divergence of $\vec{F}$ over Q.

<u>Proof</u> Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$

To show:

$$\iint_S \left(M\vec{i}\cdot\vec{n} + N\vec{j}\cdot\vec{n} + P\vec{k}\cdot\vec{n}\right) dS$$

$$= \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dV$$

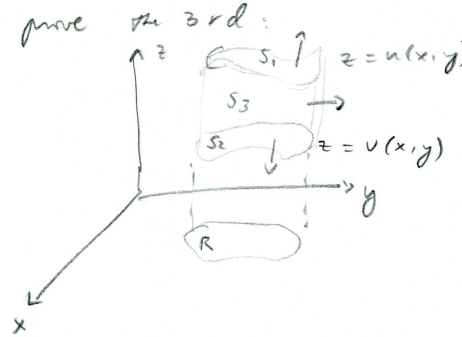
It is enough to show:

$$\iint_S M\vec{i}\cdot\vec{n}\,dS = \iiint_Q \frac{\partial M}{\partial x}\,dV$$

$$\iint_S N\vec{j}\cdot\vec{n}\,dS = \iiint_Q \frac{\partial N}{\partial y}\,dV$$

$$\iint_S P\vec{k}\cdot\vec{n}\,dS = \iiint_Q \frac{\partial P}{\partial z}\,dV$$

To prove the 3rd:



On S_3, the $\vec{k}$ component of $\vec{n} = 0 \Rightarrow \vec{k}\cdot\vec{n} = 0$

$$\Rightarrow \iint_{S_3} P\vec{k}\cdot\vec{n}\,dS = 0$$

Write

$$\iint_S P\vec{k}\cdot\vec{n}\,dS = \iint_{S_1} P\vec{k}\cdot\vec{n}\,dS + \iint_{S_2} P\vec{k}\cdot\vec{n}\,dS$$

S_1: let $g_1(x,y,z) = z - u(x,y)$

then $$\vec{n} = \frac{\nabla g_1(x,y,z)}{\|\nabla g_1(x,y,z)\|} = \frac{-u_x(x,y)\vec{i} - u_y(x,y)\vec{j} + \vec{k}}{\sqrt{[u_x(x,y)]^2 + [u_y(x,y)]^2 + 1}}$$

$$\Rightarrow \vec{k}\cdot\vec{n} = \frac{1}{\sqrt{[u_x(x,y)]^2 + [u_y(x,y)]^2 + 1}}$$

$$\Rightarrow \iint_{S_1} P\vec{k}\cdot\vec{n}\,dS = \iint_R P(x,y,u(x,y))\,dA$$

S_2: let $g_2(x,y,z) = z - v(x,y)$

then $$\vec{n} = -\frac{\nabla g_1(x,y,z)}{\|\nabla g_1(x,y,z)\|} = \frac{+v_x(x,y)\vec{i} + v_y(x,y)\vec{j} - \vec{k}}{\sqrt{[v_x(x,y)]^2 + [v_y(x,y)]^2 + 1}}$$

$$\Rightarrow \iint_{S_2} P\vec{k}\cdot\vec{n}\,dS = -\iint_R P(x,y,v(x,y))\,dA$$

$$\iint_S P\vec{k}\cdot\vec{n}\,dS = \iint_R \left[P(x,y,u(x,y)) - P(x,y,v(x,y)) \right] dA$$

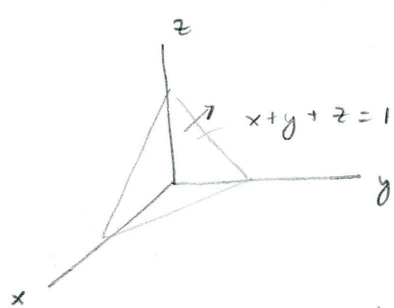
$$= \iint_R \left[\int_{v(x,y)}^{u(x,y)} \frac{\partial P}{\partial z}\, dz \right] dA$$

$$= \iiint_Q \frac{\partial P}{\partial z}\, dV$$

Ex. #2 $\vec{F} = y^3 e^z \vec{i} - xy\vec{j} + x \arctan y\, \vec{k}$

S : surface of region bounded by the coordinate planes & the plane $x+y+z=1$

Sol



$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}\left(y^3 e^z\right) + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}\left(x \arctan y\right)$$

$$= 0 - x + 0 = -x$$

By div thm,

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_Q \overset{\operatorname{div}\vec{F}\,dV}{-x\,dV}$$

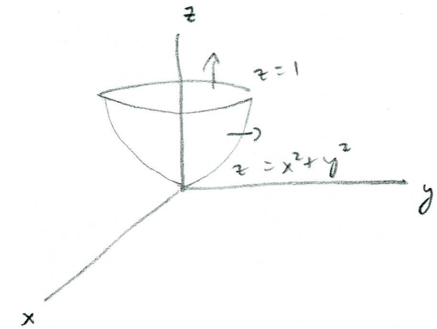
$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} -x\, dz\, dy\, dx$$

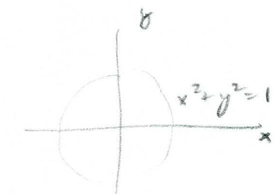
$$= -\frac{1}{24}$$

Ex #4 $\vec{F} = 2xy\vec{i} + z\cosh x\,\vec{j} + (z^2 + y\sin^{-1}x)\vec{k}$

S: $z = x^2+y^2$, $z=1$

Sol





$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(z\cosh x) + \frac{\partial}{\partial z}(z^2 + y\sin^{-1}x)$$

$$= 2y + 0 + 2z = 2(y+z)$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_Q \operatorname{div}\vec{F}\,dV$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{1} 2(y+z)\,dz\,dy\,dx$$

... cylindrical:

$$\int_0^{2\pi}\int_0^1\int_{r^2}^1 2(r\sin\theta + z)\, r\, dz\, dr\, d\theta$$

$$= \frac{2\pi}{3}$$