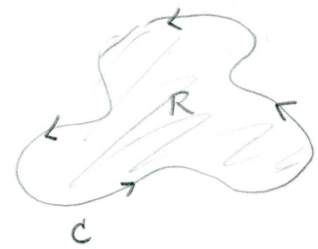
18.4. Green's Theorem

$$\oint_C M(x,y)\,dx + N(x,y)\,dy$$

- line integral along a simple closed curve C once in the positive direction



Green's Theorem Let C be a piecewise-smooth simple closed curve, and let R be the region consisting of C and its interior. If M & N are continuous functions that have continuous 1st partial derivatives throughout an open region D containing R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Proof for a special case:

$$R = R_x = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

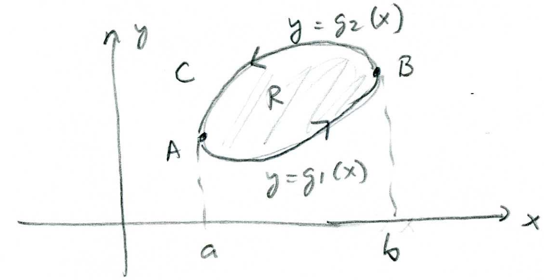
$$R = R_y = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

where g_1, g_2, h_1, h_2 are smooth functions

To show:

(1) $\displaystyle \oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dA$

(2) $\displaystyle \oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dA$



$$\oint_C M\,dx = \int_{C_1} M(x,y)\,dx + \int_{C_2} M(x,y)\,dx$$

$$= \int_a^b M(x, g_1(x))\,dx + \int_b^a M(x, g_2(x))\,dx$$

$$= \int_a^b M(x, g_1(x))\,dx - \int_a^b M(x, g_2(x))\,dx$$

Now, $\displaystyle \iint_R \frac{\partial M}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y}\,dy\,dx$

$$= \int_a^b \Big[M(x,y)\Big]_{g_1(x)}^{g_2(x)}\,dx$$

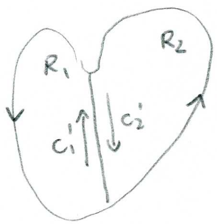
$$= \int_a^b \Big[M(x, g_2(x)) - M(x, g_1(x))\Big]\,dx$$

Thus, (1) is true.

Similarly, (2) is true.

Remark Green's theorem may be extended to a region R where part of the boundary consists of horizontal or vertical line segments. Then GT may be extended to the case where R is a finite union of such regions.

ex:



$R = R_1 \cup R_2$

$$\iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy\, dx = \oint_{C_1 \cup c_1'} M\,dx + N\,dy$$

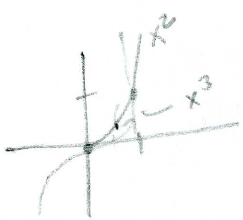
$$\iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy\, dx = \oint_{C_2 \cup c_2'} M\,dx + N\,dy$$

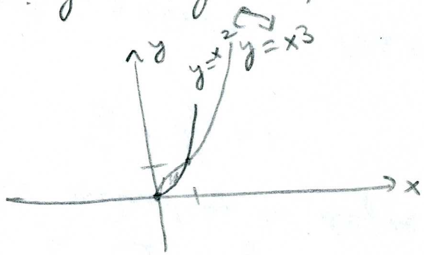
$$\Rightarrow \iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy\, dx = \oint_{C_1 \cup C_2} M\,dx + N\,dy$$

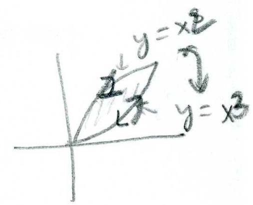
Ex # 2 $\oint_C (x+y^2)\,dx + (1+x^2)\,dy$

$C: y = x^3,\ y = x^2,\ 0 \le x \le 1$

Sol







$$\oint_C (x+y^2)\,dx + (1+x^2)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(1+x^2) - \frac{\partial}{\partial y}(x+y^2) \right] dA$$

$$= \int_0^1 \int_{x^3}^{x^2} (2x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[2xy - y^2\right]_{x^3}^{x^2} dx = \int_0^1 (2x^3 - x^4 - 2x^4 + x^6)\,dx$$

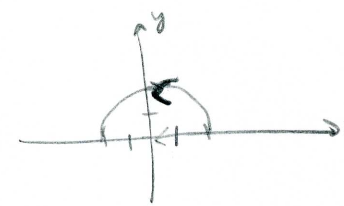
$$= \left[\frac{3x^5}{5} + \frac{x^7}{7} + \frac{2x^4}{4} \right]_0^1 = -\frac{3}{5} + \frac{1}{7} + \frac{1}{2} = \frac{-42+10+35}{70}$$

$$= +\frac{3}{70}$$

Ex # 6 $\oint_C y^2\,dx + x^2\,dy$;

C: semicircle $y = \sqrt{4-x^2}$ & x-axis

Sol



$$\oint_C y^2\,dx + x^2\,dy = \iint_R \left[\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} y^2 \right] dA$$

$$= \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} (2x - 2y)\,dy\,dx \dashrightarrow \text{polar}$$

$$= 2\int_0^{\pi} \int_0^2 (r\cos\theta - r\sin\theta)\, r\,dr\,d\theta$$

$$= 2\int_0^{\pi} \left. \frac{r^3}{3} \right]_0^2 (\cos\theta - \sin\theta)\,d\theta = 2\int_0^{\pi} \frac{8}{3}(\cos\theta - \sin\theta)\,d\theta$$

$$= \frac{16}{3}\left[\sin\theta + \cos\theta\right]_0^{\pi} = \frac{16}{3}(-1-1) = \frac{-32}{3}$$