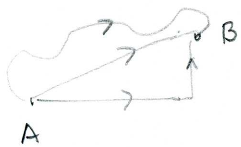
18.3. Independence of Path



paths from A to B

<u>Assume</u> : all regions are connected, i.e., any 2 pts in the region can be joined by a piecewise-smooth curve that lies in the region

<u>Theorem</u> If $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ is continuous on an open connected region D, then

$\int_C \vec{F}\cdot d\vec{r}$ is independent of path

$\Leftrightarrow$ $\vec{F}$ is conservative; i.e., $\vec{F}(x,y) = \nabla f(x,y)$ for some scalar function f.

<u>Proof</u> (" $\Rightarrow$ ")

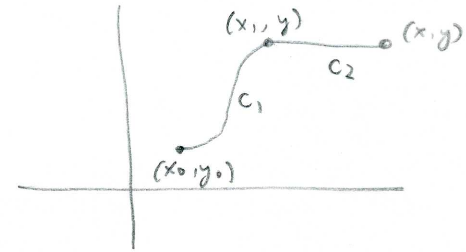
Let $\int_C \vec{F}\cdot d\vec{r}$ be independent of path in D.

Consider (x_0,y_0), a fixed pt in D, and define f by

$$f(x,y) = \int_{(x_0,y_0)}^{(x,y)} \vec{F}\cdot d\vec{r}$$

for every pt (x,y) in D. To show that $\vec{F}(x,y) = \nabla f(x,y)$.

Choose a circle in D with center (x,y) and let (x_1,y) be a point within the circle s.t. $x_1 \neq x$.



$$f(x,y) = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \int_{(x_0,y_0)}^{(x_1,y)} \vec{F}\cdot d\vec{r} + \int_{(x_1,y)}^{(x,y)} \vec{F}\cdot d\vec{r}$$

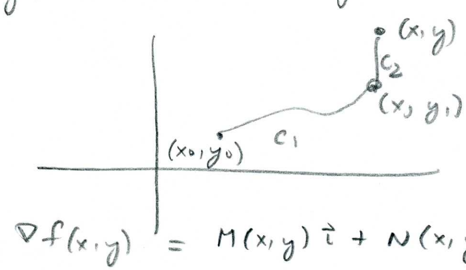
$$\frac{\partial f(x,y)}{\partial x} = 0 + \frac{\partial}{\partial x}\int_{(x_1,y)}^{(x,y)} \vec{F}\cdot d\vec{r}$$

$$= \frac{\partial}{\partial x}\int_{(x_1,y)}^{(x,y)} M\,dx + \cancel{N\underbrace{dy}_{=0}}^{\,0}$$

$$= \frac{\partial}{\partial x}\underbrace{\int_{(x_1,y)}^{(x,y)} M(x,y)\,dx}_{\text{function of } x} \quad (y \text{ fixed})$$

$$\Rightarrow \frac{\partial f(x,y)}{\partial x} = M(x,y)$$

Similarly, if we choose the path



$$\Rightarrow \frac{\partial f(x,y)}{\partial y} = N(x,y)$$

Thus, $\nabla f(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j} = \vec{F}(x,y)$.

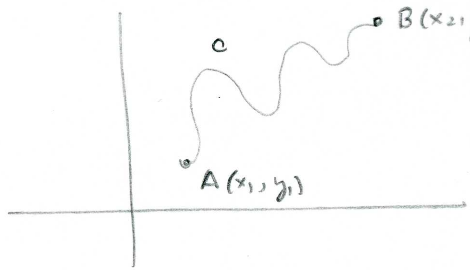
("$\Leftarrow$") Let $\vec{F}$ be conservative, i.e.,

$$\vec{F}(x,y) = \nabla f(x,y) \quad \text{for some } f$$

Then

$$M(x,y)\vec{i} + N(x,y)\vec{j} = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$$

$$\Rightarrow M(x,y) = f_x(x,y) \quad \text{and} \quad N(x,y) = f_y(x,y)$$



$$\int_C \vec{F}\cdot d\vec{r} = \int_C M(x,y)\,dx + N(x,y)\,dy$$

$$= \int_C f_x(x,y)\,dx + f_y(x,y)\,dy$$

If C : $x = g(t)$, $y = h(t)$; $t_1 \le t \le t_2$

then

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t_1}^{t_2} [f_x(g(t), h(t))\, g'(t) + f_y(g(t), h(t))\, h'(t)]\, dt$$

apply chain rule & fundamental theorem of calculus,

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t_1}^{t_2} \frac{d}{dt}[f(g(t), h(t))]\, dt$$

$$= f(g(t_2), h(t_2)) - f(g(t_1), h(t_1))$$

$$= f(x_2, y_2) - f(x_1, y_1) = \Big[f(x,y)\Big]_{(x_1,y_1)}^{(x_2,y_2)}$$

Thus, $\int_C \vec{F}\cdot d\vec{r}$ depends only on the coordinates of A & B, not on the path C $\Rightarrow$ $\int_C \vec{F}\cdot d\vec{r}$ is independent of path. ▨

__Theorem__ Let $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ be continuous on an open connected region D, and let C be a piecewise-smooth curve in D with endpoints $A(x_1,y_1)$ & $B(x_2,y_2)$. If $F(x,y) = \nabla f(x,y)$, then

$$\int_C M(x,y)\,dx + N(x,y)\,dy = \int_{(x_1,y_1)}^{(x_2,y_2)} \vec{F}\cdot d\vec{r}$$

$$= \Big[f(x,y)\Big]_{(x_1,y_1)}^{(x_2,y_2)}$$

__Remark__ $\left\{ \begin{array}{l} \int_C \vec{F}\cdot d\vec{r} \text{ independent of path} \\ (x_1,y_1) = (x_2,y_2) \end{array} \right.$

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = 0$ for every simple closed curve C.

Exercises on showing independence of path:

Ex #2 $\vec{F}(x,y) = (6xy^2 + 2y)\vec{i} + (6x^2y + 2x)\vec{j}$

__Sol__

To find $f(x,y)$ s.t.

$$f_x(x,y)\vec{i} + f_y(x,y)\vec{j} = (6xy^2 + 2y)\vec{i} + (6x^2y + 2x)\vec{j}$$

or $f_x(x,y) = 6xy^2 + 2y$ and $f_y(x,y) = 6x^2y + 2x$

$\Rightarrow f(x,y) = 3x^2y^2 + 2xy + g(y)$ (integrate f_x wrt x)

$f_y(x,y) = 6x^2y + 2x + g'(y) \Rightarrow g'(y) = 0$

$g(y) = c$

Thus, $f(x,y) = 3x^2y^2 + 2xy + c$