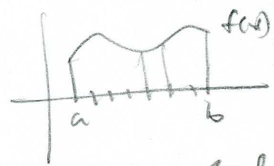
18.2. Line Integrals

ch 5: $\int_a^b f(x)\,dx$



integral of f over a closed interval $[a,b]$

line integral
- integral of a function of several variables along (or over) curves in 2D or 3D

Recall:

<u>smooth</u> plane curve C has parametrization

$x = g(t)$, $y = h(t)$; $a \le t \le b$

s.t. g' & h' are continuous and not simultaneously 0 on $[a,b]$

<u>smooth</u> space curve C ...

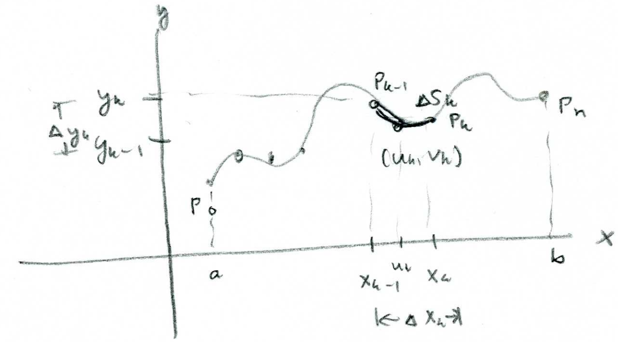
$x = g(t)$, $y = h(t)$, $z = k(t)$

<u>positive direction</u> ... increasing values of t

<u>piecewise smooth</u> ...

Let $f = f(x,y)$ continuous on a region D containing a smooth curve C : $x = g(t)$, $y = h(t)$; $a \le t \le b$

To define 3 different integrals of f along C:



$$\int_C f(x,y)\,ds = \lim_{\|P\|\to 0} \sum_k f(u_k,v_k)\,\Delta S_k$$

$$\int_C f(x,y)\,dx = \lim_{\|P\|\to 0} \sum_k f(u_k,v_k)\,\Delta x_k$$

$$\int_C f(x,y)\,dy = \lim_{\|P\|\to 0} \sum_k f(u_k,v_k)\,\Delta y_k$$

$$\Rightarrow \int_C f(x,y)\,ds = \int_a^b f(g(t),h(t))\,\sqrt{(g'(t))^2 + (h'(t))^2}\,dt$$

$$\int_C f(x,y)\,dx = \int_a^b f(g(t),h(t))\,g'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(g(t),h(t))\,h'(t)\,dt$$

<u>Note</u>: can be extended to $C = C_1 + C_2 + \dots + C_n$

i.e. C piecewise smooth

Ex # 2 $\quad f(x,y) = xy^{2/5}$, $x = \frac{1}{2}t$, $y = t^{5/2}$, $0 \le t \le 1$

Sol

a) $$\int_C f(x,y)\,ds = \int_0^1 \left(\tfrac{1}{2}t\right)\left(t^{5/2}\right)^{2/5} \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{5}{2}t^{3/2}\right)^2}\,dt$$

$$= \int_0^1 \frac{1}{2}t^2 \sqrt{\frac{1}{4} + \frac{25}{4}t^3}\,dt$$

$$= \frac{1}{4}\int_0^1 t^2 (1+25t^3)^{1/2}\,dt$$

$$= \frac{1}{4}(1+25t^3)^{3/2}\cdot\frac{2}{3}\cdot\frac{1}{75}\Big]_0^1$$

$$= \frac{1}{6\cdot 75}\left[(26)^{3/2} - 1\right] \approx 0.29$$

b) $$\int_C f(x,y)\,\underline{dx} = \int_0^1 \frac{1}{2}t^2\cdot\underline{\frac{1}{2}\,dt}$$

$$= \frac{1}{4}\frac{t^3}{3}\Big]_0^1 = \frac{1}{12}$$

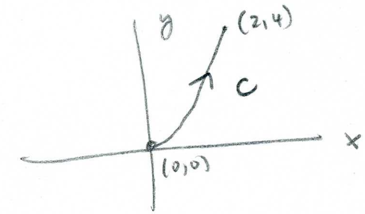
c) $$\int_C f(x,y)\,\underline{dy} = \int_0^1 \frac{1}{2}t^2\ \underline{\frac{5}{2}t^{3/2}}\,dt$$

$$= \frac{5}{4}\int_0^1 t^{7/2}\,dt = \frac{5}{4}t^{9/2}\cdot\frac{2}{9}\Big]_0^1$$

$$= \frac{5}{18}$$

Ex2 p974 $\quad$ Evaluate $\int_C xy^2\,dx$ and $\int_C xy^2\,dy$ if C is the portion of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$.

Sol



let C: $x = t$, $y = t^2$; $0 \le t \le 2$

$dx = dt \quad dy = 2t\,dt$

$$\int_C xy^2\,dx = \int_0^2 t(t^2)^2\,dt = \int_0^2 t^5\,dt = \frac{t^6}{6}\Big]_0^2 = \frac{32}{3}$$

$$\int_C xy^2\,dy = \int_0^2 t(t^2)^2\,2t\,dt = 2\int_0^2 t^6\,dt = \frac{2t^7}{7}\Big]_0^2 = \frac{256}{7}$$

Note : C: $x = t$, $y = g(t)$; $a \le t \le b$

$$\int_C f(x,y)\,dx = \int_a^b f(t, g(t))\,dt = \int_a^b f(x, g(x))\,dx$$

$\Rightarrow$ $$\int_C f(x,y)\,dy = \int_a^b f(t,g(t))\,g'(t)\,dt$$

$$= \int_a^b f(x, g(x))\,g'(x)\,dx$$

$\Rightarrow$ we may bypass the parametric equations

So for Ex 2, $$\int_C xy^2\,dx = \int_0^2 x(x^4)\,dx = \int_0^2 x^5\,dx = \frac{32}{3}$$

$$\int_C xy^2\,dy = \int_0^2 x(x^4)\,2x\,dx = \int_0^2 2x^6\,dx = \frac{256}{7}$$