Math 267 Exam 4 Notes

Let f be a continuous function of two variables such that $f(x,y) \geq 0$ for every (x,y) in a region R. The **volume** V of the solid that lies under the graph of $z = f(x,y)$ and over R is $V = \iint\limits_R f(x,y)\,dA$.

(i) $\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ **(ii)** $\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

DOUBLE INTEGRALS IN POLAR COORDINATES $\iint\limits_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$

<u>Change of variables formula</u> $\iint\limits_R f(x,y)\,dy\,dx = \iint\limits_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

SURFACE AREA $A = \iint\limits_R \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\,dA$

TRIPLE INTEGRALS $\iiint\limits_Q f(x,y,z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{k_1(x,z)}^{k_2(x,z)} f(x,y,z)\,dy\,dz\,dx$

<u>Mass density</u> $\delta(x,y,z) = \lim\limits_{\|\Delta V_k\| \to 0} \dfrac{\Delta m_k}{\Delta V_k}$

<u>Mass of a solid</u> $m = \iiint\limits_Q \delta(x,y,z)\,dV$

<u>Mass of a lamina</u> $m = \iint\limits_R \delta(x,y)\,dA$

MOMENTS AND CENTER OF MASS

(i) $m = \iint\limits_R \delta(x,y)\,dA$

(ii) $M_x = \iint\limits_R y\,\delta(x,y)\,dA, \quad M_y = \iint\limits_R x\,\delta(x,y)\,dA$

(iii) $\bar{x} = \dfrac{M_y}{m} = \dfrac{\iint\limits_R x\,\delta(x,y)\,dA}{\iint\limits_R \delta(x,y)\,dA}, \quad \bar{y} = \dfrac{M_x}{m} = \dfrac{\iint\limits_R y\,\delta(x,y)\,dA}{\iint\limits_R \delta(x,y)\,dA}$

Moments of inertia of a lamina

$I_x = \iint\limits_R y^2\delta(x,y)\,dA \qquad I_y = \iint\limits_R x^2\delta(x,y)\,dA \qquad I_o = \iint\limits_R \left(x^2+y^2\right)\delta(x,y)\,dA$

<u>Moments and center of mass in three dimensions</u>

(i) $m = \iiint\limits_Q \delta(x,y,z)\,dV$

(ii) $M_{xy} = \iiint\limits_Q z\,\delta(x,y,z)\,dV \qquad M_{xz} = \iiint\limits_Q y\,\delta(x,y,z)\,dV \qquad M_{yz} = \iiint\limits_Q x\,\delta(x,y,z)\,dV$

(iii) $\bar{x} = \dfrac{M_{yz}}{m};\ \bar{y} = \dfrac{M_{xz}}{m};\ \bar{z} = \dfrac{M_{xy}}{m}$

<u>Moments of Inertia of solids</u>

$I_z = \iiint\limits_Q \left(x^2+y^2\right)\delta(x,y,z)\,dV \qquad I_x = \iiint\limits_Q \left(y^2+z^2\right)\delta(x,y,z)\,dV \qquad I_y = \iiint\limits_Q \left(x^2+z^2\right)\delta(x,y,z)\,dV$

Theorem of Pappus Let R be a region in a plane that lies entirely on one side of a line l in the plane. If R is revolved once about l, the volume of the resulting solid is the product of the area of R and the distance traveled by the centroid of R.

CYLINDRICAL COORDINATES

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \tan\theta = \frac{y}{x},$$
$$r^2 = x^2 + y^2, \qquad z = z$$

$$\iiint_Q f(r,\theta,z)\,dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r,\theta,z)\, r\, dz\, dr\, d\theta$$

SPHERICAL COORDINATES

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$
$$\rho^2 = x^2 + y^2 + z^2$$

$$\iiint_Q f(\rho,\phi,\theta)\,dV = \int_m^n \int_c^d \int_a^b f(\rho,\phi,\theta)\,\rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$

CHANGE OF VARIABLES AND JACOBIANS

If $x = f(u,v)$ and $y = g(u,v)$, then the **Jacobian** of x and y with respect to u and v is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

If $x = f(u,v)$, $y = g(u,v)$ is a transformation of coordinates, then

$$\iint_R F(x,y)\,dx\,dy = \pm\iint_S F(f(u,v),g(u,v))\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv\,.$$

If $x = f(u,v)$, $y = g(u,v)$ is a transformation of coordinates and if the Jacobian does not change sign in S, then

$$\iint_R F(x,y)\,dx\,dy = \iint_S F(f(u,v),g(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

If $x = f(u,v,w)$, $y = g(u,v,w)$, and $z = h(u,v,w)$, then the **Jacobian** of x, y, and z with respect to u, v, and w is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

If $x = f(u,v,w)$, $y = g(u,v,w)$, $z = h(u,v,w)$ is a transformation of coordinates from a region S in a uvw-coordinate system onto a region R in an xyz-coordinate system and if the Jacobian does not change sign in S, then

$$\iiint_R F(x,y,z)\,dx\,dy\,dz = \iiint_S G(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw\,,$$

where $G(u,v,w)$ is the expression obtained by substituting for x, y, and z in $F(x,y,z)$.