Math 267 Exam 3 Notes

Ellipsoid	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$	
Hyperboloid of one sheet	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$	axis is z – axis
Hyperboloid of two sheets	$-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$	axis is z – axis
Cone	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0$	axis is z – axis
Paraboloid	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=cz$	axis is z – axis
Hyperbolic Paraboloid	$\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=cz$	axis is z – axis

A **surface of revolution** is obtained by revolving a plane curve C about a line (the axis of revolution) in the plane. To obtain the equation of the surface obtained when the curve $f(x,y)=0$ in the xy-plane is revolved about the x-axis, replace y by $\sqrt{y^2+z^2}$, etc.

FUNCTIONS OF TWO OR MORE VARIABLES

Limits : *2-path rule* **continuity**: determine domain in $\mathbb{R}^2$ or $\mathbb{R}^3$

increment $\Delta w = f(x+\Delta x, y+\Delta y) - f(x,y)$.

$$\Delta w = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \in_1 \Delta x + \in_2 \Delta y ,$$

where $\in_1$ and $\in_2$ are functions of Δx and Δy that have the limit 0 as $(\Delta x,\Delta y)\to(0,0)$.

differential dw of the dependent variable w is $dw = f_x(x,y)dx + f_y(x,y)dy = \dfrac{\partial w}{\partial x}dx + \dfrac{\partial w}{\partial y}dy$

CHAIN RULE If $w=f(u,v)$, with $u=g(x,y), v=h(x,y),$ and if $f, g,$ and h are differentiable, then

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x};\ \frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

If an equation $F(x,y)=0$ determines, *implicitly*, a differentiable function f of one variable x such that $y=f(x)$, then $\dfrac{dy}{dx}=-\dfrac{F_x(x,y)}{F_y(x,y)}$.

If an equation $F(x,y,z)=0$ determines an *implicit* differentiable function f of two variables x and y such that $z=f(x,y)$ for every (x,y) in the domain of f, then

$$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)},\qquad \frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)}.$$

Let $w=f(x,y)$, and let $\vec{u}=u_1\vec{i}+u_2\vec{j}=\langle u_1,u_2\rangle$ be a unit vector. The **directional derivative of** f **at** $P(x,y)$ **in the direction of** $\vec{u}$, denoted by $D_{\vec{u}}f(x,y)$, is

$$D_{\vec{u}}f(x,y)=\lim_{s\to 0}\frac{f(x+su_1,y+su_2)-f(x,y)}{s}=f_x(x,y)u_1+f_y(x,y)u_2=\nabla f(x,y)\cdot\vec{u}$$

The **gradient** of *f* is $\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$ (orthogonal to level curve/surface)
Let *f* be a function of two variables that is differentiable at the point $P(x,y)$.

(i) The maximum value of $D_{\vec{u}} f(x,y)$ at $P(x,y)$ is $\|\nabla f(x,y)\|$.
(ii) The maximum rate of increase of $f(x,y)$ at $P(x,y)$ occurs in the direction of $\nabla f(x,y)$.
(iii) The minimum value of $D_{\vec{u}} f(x,y)$ at $P(x,y)$ is $-\|\nabla f(x,y)\|$.
(iv) The minimum rate of increase (or maximum rate of decrease) of $f(x,y)$ at $P(x,y)$ occurs in the direction of $-\nabla f(x,y)$.

An **equation for the tangent plane** to the graph of $F(x,y,z)=0$ at the point $P_0(x_0,y_0,z_0)$ is
$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$

The **normal line** to *S* at P_0 is parallel to the vector $\nabla F(x_0,y_0,z_0)$.

Let *f* be a function of two variables. A pair (a,b) is a **critical point** of *f* if either

(i) $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or
(ii) $f_x(a,b)$ or $f_y(a,b)$ does not exist.

discriminant *D* of *f* is given by $D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(x,y)f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2$

<u>Test for local extrema</u>

Let *f* be a function of two variables that has continuous second partial derivatives throughout an open disk *R* containing (a,b). If $f_x(a,b) = f_y(a,b) = 0$ and $D(a,b)>0$, then

(i) $f_{xx}(a,b) <0 \Rightarrow f(a,b)$ is a local maximum
(ii) $f_{xx}(a,b) >0 \Rightarrow f(a,b)$ is a local minimum

If $D(a,b) < 0$, then $(a,b,f(a,b))$ is a **saddle point**.

To get extrema over a closed and bounded region: get local extrema and test extrema the boundary.

Constrained Optimization
Suppose *f* and *g* are functions of two variables that have continuous first partial derivatives, and that $\nabla g \neq \vec{0}$ throughout a region of the *xy* – plane. If *f* has an extremum $f(x_0,y_0)$ subject to the constraint $g(x,y) = 0$, then there is a real number λ such that $\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$.
λ is called a **Lagrange multiplier**.

Technique: Solve the system

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y)=0 \end{cases}$$

Applications:
1. rate of change
2. approximations using differentials
3. related rates
4. optimization