Math 267 Exam 2 Notes

vector-valued function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} = \langle f(t), g(t), h(t) \rangle$ for every t in $D \subseteq \mathbb{R}$ defines a **space curve**

twisted cubic $x = at$, $y = bt^2$, $z = ct^3$, where *a, b,* and *c* are nonzero constants.

circular helix $\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j} + bt\,\vec{k}$ for $t \geq 0$ and positive constants *a* and *b*

length of a space curve $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

limit $\lim_{t\to a} \vec{r}(t) = \left[\lim_{t\to a} f(t)\right]\vec{i} + \left[\lim_{t\to a} g(t)\right]\vec{j} + \left[\lim_{t\to a} h(t)\right]\vec{k}$, provided *f, g,* and *h* have limits as $t \to a$

$\vec{r}(t)$ is **continuous** at *a* if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

derivative of $\vec{r}$ $\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$; $\vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$

differentiation formulas If $\vec{u}$ and $\vec{v}$ are differentiable vector-valued functions and *c* is a scalar, then

(i) $D_t\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}\,'(t) + \vec{v}\,'(t)$

(ii) $D_t\left[c\,\vec{u}(t)\right] = c\,\vec{u}\,'(t)$

(iii) $D_t\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}(t)\cdot\vec{v}\,'(t) + \vec{u}\,'(t)\cdot\vec{v}(t)$

(iv) $D_t\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}(t)\times\vec{v}\,'(t) + \vec{u}\,'(t)\times\vec{v}(t)$

$\vec{r}\,'(t)$ is the **tangent vector** to *C* at *P.*

If $\vec{r}$ is differentiable and $\|\vec{r}(t)\|$ is constant, then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$ for every *t* in the domain of $\vec{r}\,'$.

definite integral of $\vec{r}$ **from** a **to** b is $\int_a^b r(t)dt = \left[\int_a^b f(t)dt\right]\vec{i} + \left[\int_a^b g(t)dt\right]\vec{j} + \left[\int_a^b h(t)dt\right]\vec{k}$

$\int_a^b \vec{r}(t)dt = \vec{R}(t)\Big]_a^b = \vec{R}(b) - \vec{R}(a)$, where $\vec{R}(t)$ is an **antiderivative** of $\vec{r}(t)$.

position vector $\vec{r}(t) = x\vec{i} + y\vec{j} = f(t)\vec{i} + g(t)\vec{j}$,

velocity: $\vec{v}(t) = \vec{r}\,'(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$

speed: $v(t) = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

acceleration: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j}$

<u>Motion around a circle of radius *k* at a constant speed *v*</u>

$v = \omega k$ $\|\vec{a}(t)\| = \frac{v^2}{k}$

<u>Newton's Second Law of Motion</u> $\vec{F} = m\vec{a}$

Projectile Motion

initial velocity $\vec{v}_0$ from a point h_0 feet above the ground, $\vec{v}_0$ makes an angle α with the horizontal; only force acting is g

$$r(t) = \left(-\tfrac{1}{2}gt^2 + h_0\right)\vec{j} + \vec{v}_0 t \quad ; \quad \vec{v}_0 = \langle v_0\cos\alpha, v_0\sin\alpha\rangle$$

unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ **principal unit normal vector** $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$

A variable s is an **arc length parameter** for a plane curve C if there is a parametrization

$$x = f(s), \quad y = g(s)$$

such that s is the length of C from a fixed point A on the curve to the point $P(x,y)$.
The parametric equations are called an **arc length parametrization** for C.

Curvature - curve C defined by:

(1) arc length parametrization $x = f(s), \quad y = g(s)$, and let θ be the angle between $\vec{T}(s)$ and $\vec{i}$

$$K = \left|\frac{d\theta}{ds}\right|$$

(2) $y = f(x)$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}}.$$

(3) parametrization $x = f(t), y = g(t)$

$$K = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{\left[(f'(t))^2 + (g'(t))^2\right]^{3/2}}.$$

The circle of radius $\rho = 1/K$ whose center lies on the concave side of C and has the same tangent line at P as C is the **circle of curvature** for P. Its radius ρ and center are the **radius of curvature** and **center of curvature**, respectively, for P.

curvature in 3-space

(4) C has an arc length parametrization $x = f(s), \quad y = g(s), \quad z = h(s)$ and let $\vec{T}(s) = \vec{r}\,'(s)$

$$K = \|\vec{T}\,'(s)\|.$$

If $\vec{T}(s)$ and $\vec{N}(s)$ are the unit tangent and principal unit normal vectors and if K is the curvature of C at the point corresponding to s, then $\vec{v}(t) = \dfrac{ds}{dt}\vec{T}(s)$ and $\vec{a}(t) = \dfrac{d^2s}{dt^2}\vec{T}(s) + K\left(\dfrac{ds}{dt}\right)^2\vec{N}(s)$

Tangential component of acceleration $a_{\vec{T}} = \dfrac{d^2s}{dt^2} = \dfrac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\|\vec{r}\,'(t)\|}$

Normal component of acceleration $a_{\vec{N}} = K\left(\dfrac{ds}{dt}\right)^2 = \dfrac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|} = \sqrt{\|\vec{a}\|^2 - a_{\vec{T}}^2}$

(5) C has the parametrization $x = f(t), \quad y = g(t), \quad z = h(t)$

$$K = \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{a_{\vec{N}}}{\|r'(t)\|^2}$$