Linear Algebra

Linear Transformations

Topics

• Preliminaries
• Definition and Examples
• Kernel and Range of a Linear Transformation
• Matrix of a Linear Transformation
• Vector Spaces of Matrices and Linear Transformations
• Similarity
• Homogeneous Coordinates
Linear Algebra

Linear Transformations

Topics

• Preliminaries
• Definition and Examples
• Kernel and Range of a Linear Transformation
• Matrix of a Linear Transformation
• Vector Spaces of Matrices and Linear Transformations
• Similarity
• Homogeneous Coordinates
Linear Algebra

Linear Transformations

Preliminaries

• Have pursued the following generalizations
  Vectors in R^2 & R^3 ⇒ Vector Spaces
  Dot Product in R^2 & R^3 ⇒ Inner Product

• Will now look at another generalization
  Matrices ⇒ Linear Transformations
Linear Algebra

Linear Transformations

Topics

• Preliminaries
• \textit{Definition and Examples}
• Kernel and Range of a Linear Transformation
• Matrix of a Linear Transformation
• Vector Spaces of Matrices and Linear Transformations
• Similarity
• Homogeneous Coordinates
• **Defn** - Let \( V \) and \( W \) be vector spaces. A function \( L: V \rightarrow W \) is called a **linear transformation** of \( V \) into \( W \) if

a) \( L(u + v) = L(u) + L(v) \)

b) \( L(cu) = cL(u) \) for \( u \in V \) and real \( c \)

• If \( V = W \), then \( L \) is called a **linear operator**

• Note: An \( m \times n \) matrix takes a vector in \( \mathbb{R}^n \) and maps it to a vector in \( \mathbb{R}^m \), so it can be viewed as a function from \( \mathbb{R}^n \) to \( \mathbb{R}^m \)
Example

• Define a mapping \( L: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) as \( L \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \)

To verify, let \( \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \), \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \) be arbitrary

\[
L(\mathbf{u} + \mathbf{v}) = L \left( \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \right) = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} = L \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + L \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{u} + \mathbf{v}
\]

\[
L(c\mathbf{u}) = L \left( \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} \right) = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = cL(\mathbf{u})
\]
Example

Let $K(x,y)$ be continuous in $x$ and $y$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Define $L: C\left[0,1\right] \rightarrow C\left[0,1\right]$ as

$$L(f) = \int_{0}^{1} K(x,y) f(y) \, dy$$

From the properties of integrals, conditions (a) and (b) hold.
Example

• Define a mapping $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as $L\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = r \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$L$ is a linear operator on $\mathbb{R}^3$. If $r > 1$, it is called a dilation. If $0 < r < 1$, it is called a contraction. General term is scaling.
Example

- Consider the vector space $C^\infty [0,1]$ of infinitely differentiable functions defined on the interval $[0,1]$. Define a mapping $L: C^\infty [0,1] \rightarrow C^\infty [0,1]$ by

$$L(f) = f'$$

$L$ is a linear operator on $C^\infty [0,1]$
Example

• Define a mapping $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as

$$L \left( \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$L$ is a linear transformation. More generally, if $A$ is an $m \times n$ matrix, then $L(x) = Ax$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
Example

- Define a mapping $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as $L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$

This is a linear operator, which is called a reflection in the $x$-axis.
Example

- Define a mapping \( L: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) as

\[
L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

\( L \) is a linear transformation. It is a counter-clockwise rotation by the angle \( \phi \).
Example

- Define a mapping \( L: \mathbb{R}^3 \to \mathbb{R}^3 \) as

\[
L\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 + 1 \\ 2a_2 \\ a_3 \end{bmatrix}
\]

Let \( \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \), \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \)

\[
L(\mathbf{u} + \mathbf{v}) = L\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + 1 \\ 2(u_2 + v_2) \\ u_3 + v_3 \end{bmatrix}
\]

\[
L(\mathbf{u}) + L(\mathbf{v}) = \begin{bmatrix} u_1 + 1 \\ 2u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 + 1 \\ 2v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + 2 \\ 2(u_2 + v_2) \\ u_3 + v_3 \end{bmatrix}
\]
Example

- Define a mapping \( L: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) as \( L\left( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) = \begin{bmatrix} a_1^2 \\ 2a_2 \end{bmatrix} \)

Let \( \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \), \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \) be in \( \mathbb{R}^2 \)

\[
L(\mathbf{u} + \mathbf{v}) = L\left( \begin{bmatrix} (u_1 + v_1) \\ (u_2 + v_2) \end{bmatrix} \right) = \begin{bmatrix} (u_1 + v_1)^2 \\ 2(u_2 + v_2) \end{bmatrix}
\]

\[
L(\mathbf{u}) + L(\mathbf{v}) = \begin{bmatrix} u_1^2 \\ 2u_2 \end{bmatrix} + \begin{bmatrix} v_1^2 \\ 2v_2 \end{bmatrix} = \begin{bmatrix} (u_1^2 + v_1^2) \\ 2(u_2 + v_2) \end{bmatrix}
\]

\[
L(\mathbf{u} + \mathbf{v}) \neq L(\mathbf{u}) + L(\mathbf{v})
\]

So \( L \) is not a linear transformation
• **Theorem** - Let $L: V \rightarrow W$ be a linear transformation of an $n$ dimensional vector space $V$ into a vector space $W$. Let $S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \}$ be a basis for $V$. If $\mathbf{v}$ is any vector in $V$, then $L(\mathbf{v})$ is completely determined by the set of vectors \{ $L(\mathbf{u}_1)$, $L(\mathbf{u}_2)$, $\ldots$, $L(\mathbf{u}_n)$ \}

• **Proof** - Since $S$ is a basis for $V$, can express $\mathbf{v}$ as

$$\mathbf{v} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \cdots + a_n \mathbf{u}_n$$

Then

$$L(\mathbf{v}) = L(a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \cdots + a_n \mathbf{u}_n)$$

$$= L(a_1 \mathbf{u}_1) + L(a_2 \mathbf{u}_2) + \cdots + L(a_n \mathbf{u}_n)$$

$$= a_1 L(\mathbf{u}_1) + a_2 L(\mathbf{u}_2) + \cdots + a_n L(\mathbf{u}_n)$$

So $L(\mathbf{v})$ can be expressed as a combination of the vectors \{ $L(\mathbf{u}_1)$, $L(\mathbf{u}_2)$, $\ldots$, $L(\mathbf{u}_n)$ \}

QED
• **Corollary** - Let $L: V \rightarrow W$ and $T: V \rightarrow W$ be linear transformations. Let $S = \{ v_1, v_2, \ldots, v_n \}$ be a basis for $V$. If $L(v_i) = T(v_i)$ for $1 \leq i \leq n$, then $L(v) = T(v)$ for all $v \in V$, i.e. $L$ and $T$ are identical linear transformations.
Example

- Let \( L: \mathbb{R}^4 \rightarrow \mathbb{R}^2 \) be a linear transformation and let 
  \[ S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \} \]
  be a basis for \( \mathbb{R}^4 \)

  \[
  \begin{align*}
  \mathbf{v}_1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, & \mathbf{v}_2 &= \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, & \mathbf{v}_3 &= \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}, & \mathbf{v}_4 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
  \end{align*}
  \]

Let 

\[
\begin{align*}
L(\mathbf{v}_1) &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}, & L(\mathbf{v}_2) &= \begin{bmatrix} 0 \\ 3 \end{bmatrix}, & L(\mathbf{v}_3) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & L(\mathbf{v}_4) &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
\end{align*}
\]
Linear Algebra

Linear Transformations

Example (continued)

Let \( \mathbf{v} = \begin{bmatrix} 3 \\ -5 \\ -5 \\ 0 \end{bmatrix} = 2 \mathbf{v}_1 + \mathbf{v}_2 - 3 \mathbf{v}_3 + \mathbf{v}_4 \)

So \( L(\mathbf{v}) = 2L(\mathbf{v}_1) + L(\mathbf{v}_2) - 3L(\mathbf{v}_3) + L(\mathbf{v}_4) \)

\[ = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \]
• **Theorem** - Let $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and $A$ be the $m \times n$ matrix whose $j$th column is $L(e_j)$, where $\{ e_1, e_2, \ldots, e_n \}$ is the natural basis for $\mathbb{R}^n$. Then for every $x \in \mathbb{R}^n$, $L(x) = Ax$. Moreover, $A$ is the only matrix with this property.

• **Proof** - Express $x$ in terms of the natural basis as $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$. By the properties of the linear transformation and the definition of $A$:

$$L(x) = L\left(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n\right) = x_1 L(e_1) + x_2 L(e_2) + \cdots + x_n L(e_n) = \left[ L(e_1) \quad L(e_2) \quad \cdots \quad L(e_n) \right] x = Ax$$
• **Proof** (continued) -

To argue uniqueness, suppose that there is a matrix $B \neq A$ such that $L(x) = Bx$ for every $x \in \mathbb{R}^n$. Since $B \neq A$, $A$ and $B$ must differ in at least one column, call it $j$. By the definition of $A$ and $B$, $L(e_j) = Ae_j = Be_j$. $Ae_j$ is just the $j$th column of $A$, $Be_j$ is just the $j$th column of $B$, so the $j$th columns of $A$ and $B$ are the same, which is a contradiction. Therefore $A$ is unique.

QED
• **Theorem** - Let $L : V \rightarrow W$ be a linear transformation. Then

  a) $L(\mathbf{0}_V) = \mathbf{0}_W$
  
  b) $L(\mathbf{u} - \mathbf{v}) = L(\mathbf{u}) - L(\mathbf{v})$

• **Proof** - 

  a) $\mathbf{0}_V = \mathbf{0}_V + \mathbf{0}_V$ then $L(\mathbf{0}_V) = L(\mathbf{0}_V) + L(\mathbf{0}_V)$,
  
  $L(\mathbf{0}_V) - L(\mathbf{0}_V) = L(\mathbf{0}_V) + L(\mathbf{0}_V) - L(\mathbf{0}_V)$. So
  
  $\mathbf{0}_W = L(\mathbf{0}_V)$

  b) $L(\mathbf{u} - \mathbf{v}) = L(\mathbf{u} + (-1)\mathbf{v}) = L(\mathbf{u}) + (-1)L(\mathbf{v})$

  $= L(\mathbf{u}) - L(\mathbf{v})$

QED
Linear Algebra

Linear Transformations

Topics

• Preliminaries
• Definition and Examples
• *Kernel and Range of a Linear Transformation*
• Matrix of a Linear Transformation
• Vector Spaces of Matrices and Linear Transformations
• Similarity
• Homogeneous Coordinates
Defn - A linear transformation $L: V \to W$ is **one to one** if it is a one to one function, i.e. if $v_1 \neq v_2$ implies $L(v_1) \neq L(v_2)$. (Equivalently, $L$ is one to one if $L(v_1) = L(v_2)$ implies $v_1 = v_2$.)

Defn - Let $L: V \to W$ be a linear transformation. The **kernel** of $L$, ker $L$, is the subset of $V$ consisting of all $v \in V$ such that $L(v) = 0_W$

Comment - Since $L(0_V) = 0_W$, ker $L$ is not empty
Linear Algebra

Linear Transformations

Kernel and Range of a Linear Transformation

• **Theorem** - Let $L: V \rightarrow W$ be a linear transformation
  a) $\ker L$ is a subspace of $V$
  b) $L$ is one to one if and only if $\ker L = \{ 0_V \}$

• **Proof** -
  a) Use the theorem that tests for subspaces. Specifically, if $U$ is a nonempty subset of $V$, it is a subspace if $v + w \in U$ and $cv \in U$ for all $v, w \in U$ and all real $c$. So let $v, w \in \ker L$ be arbitrary. Then $L(v) = 0_w$ and $L(w) = 0_w$. Since $L$ is linear,

  $$L(v + w) = L(v) + L(w) = 0_w + 0_w = 0_w$$

  So $v + w \in \ker L$
• **Proof** (continued)

Let $v \in \ker L$ and real $c$ be arbitrary. Since $L$ is a linear transformation. $L(cv) = cL(v) = c0_w = 0_w$

So $cv \in \ker L$

b) $\implies$ Let $L$ be one to one. Let $v \in \ker L$ be arbitrary. Then $L(v) = 0_w$. Also, $L(0_v) = 0_w$. Since $L$ is one to one, $L(v) = L(0_v)$ implies $v = 0_v$

So $\ker L = \{ 0_v \}$

$\iff$ Let $\ker L = \{ 0_v \}$ and let $v, w \in V$ be such that $L(v) = L(w)$. Need to show $v = w$. Since $L$ is linear, $0_w = L(v) - L(w) = L(v - w)$. So $v - w \in \ker L$ and $v - w = 0_v$ or $v = w$. So $L$ is one to one

QED
• Note - Part (b) of the preceding theorem can be expressed as: $L$ is one to one if and only if $\dim \ker L = 0$
• **Corollary** - Let $L: V \rightarrow W$ be a linear transformation. If $L(x) = b$ and $L(y) = b$, then $x - y$ belongs to $\ker L$, i.e. any two solutions to $L(x) = b$ differ by an element of the kernel of $L$.

• **Proof** - Suppose that $L(x) = b$ and $L(y) = b$. Then $0_W = b - b = L(x) - L(y) = L(x - y)$. Therefore, $x - y$ belongs to $\ker L$. 

QED
Linear Algebra

Linear Transformations

Kernel and Range of a Linear Transformation

Example

• Define $L: P_2 \rightarrow \mathbb{R}$ as

$$L(at^2 + bt + c) = \int_0^1 (at^2 + bt + c)\,dt$$

i) Find ker $L$

ii) Find dim ker $L$

iii) Determine if $L$ is one to one

$$\int_0^1 (at^2 + bt + c)\,dt = \frac{1}{3}a + \frac{1}{2}b + c$$
Example (continued)

\[
\int_0^1 \left( at^2 + bt + c \right) \, dt = 0 \implies \frac{1}{3} a + \frac{1}{2} b + c = 0
\]

\[
\implies c = -\frac{1}{3} a - \frac{1}{2} b
\]

So \( \ker L \) consists of polynomials of the form

\[
at^2 + bt + \left( -\frac{1}{3} a - \frac{1}{2} b \right)
\]
Example (continued)

ii) \[ at^2 + bt + \left(-\frac{1}{3}a - \frac{1}{2}b\right) = a\left(t^2 - \frac{1}{3}\right) + b\left(t - \frac{1}{2}\right) \]

So the vectors \( t^2 - 1/3 \) and \( t - 1/2 \) span \( \ker L \). Can argue that they are linearly independent. So the set \{ \( t^2 - 1/3 \), \( t - 1/2 \) \} is a basis for \( \ker L \).

iii) Since \( \dim \ker L = 2 \), \( L \) is not one to one
• **Defn** - Let $L : V \to W$ be a linear transformation. The *range* of $L$, or *image* of $V$ under $L$, denoted by $\text{range } L$, consists of all vectors $w \in W$ such that $w = L(v)$ for some $v \in V$.

• **Defn** - The linear transformation $L : V \to W$ is *onto* if $\text{range } L = W$.
• **Theorem** - Let $L: V \to W$ be a linear transformation. Then range $L$ is a subspace of $W$.

• **Proof** - Let $w_1, w_2 \in \text{range } L$ be arbitrary. Then $w_1 = L(v_1)$ and $w_2 = L(v_2)$ for some $v_1, v_2 \in V$. $w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2)$. So $w_1 + w_2 \in \text{range } L$.

Let $w \in \text{range } L$ be arbitrary. Then $w = L(v)$ for some $v \in V$. Let $c$ be an arbitrary real number. $c \cdot w = c \cdot L(v) = L(cv)$. So $c \cdot w \in \text{range } L$.

$\therefore$ range $L$ is a subspace of $W$.

QED
Example

• Consider $L: \mathbb{P}_2 \rightarrow \mathbb{R}$ defined as

\[
L(at^2 + bt + c) = \int_0^1 (at^2 + bt + c) \, dt = \frac{1}{3} a + \frac{1}{2} b + c
\]

For an arbitrary real number $r$, then $0t^2 + 0t + r$ maps to $r$. So $L$ is onto and $\dim \text{range } L = 1$

• Note that $\dim \ker L + \dim \text{range } L = \dim \mathbb{P}_2$
Example

• Let $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

a) Is $L$ onto?

b) Find basis for range $L$

c) Find ker $L$

d) Is $L$ one to one?
Example (continued)

a) Let \( \mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \) be arbitrary. Find \( \mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 \) such that \( L(\mathbf{v}) = \mathbf{w} \)

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 2 \\
2 & 1 & 3
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= 
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
\Rightarrow 
\begin{bmatrix}
1 & 0 & 1 & | & a \\
1 & 1 & 2 & | & b \\
2 & 1 & 3 & | & c
\end{bmatrix}
\Rightarrow 
\begin{bmatrix}
1 & 0 & 1 & | & a \\
0 & 1 & 1 & | & b-a \\
0 & 0 & 0 & | & c-b-a
\end{bmatrix}
\]

Solution exists only if \( c - b - a = 0 \)

So, \( L \) is not onto
Example (continued)

b) Range of L is the span of \( \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \)

Can show that the first two vectors are linearly independent and the third is the sum of the first two. Alternatively, could take the transpose of the matrix and put it into row echelon form to get a basis for the row space of the transpose. Either way, basis is

\[ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

or

\[ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \]
Example (continued)

(c) Kernel of L consists of all vectors $v = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ such that $L(v) = 0$

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 2 \\
2 & 1 & 3 \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\Rightarrow
a_1 + a_2 + 2a_3 = 0 \\
2a_1 + a_2 + 3a_3 = 0
\]

Set $a_3 = r$, then $a_1 = -r$ and $a_2 = -r$.

So, all vectors in the kernel look like

\[
\begin{bmatrix}
1 \\
0 \\
-1 \\
\end{bmatrix}
\]

So, all vectors in the kernel look like

\[
\begin{bmatrix}
-1 \\
-1 \\
1 \\
\end{bmatrix}
\]

Basis for $\ker L$ is

\[
\begin{bmatrix}
-1 \\
1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 \\
1 \\
\end{bmatrix}
\]
Example (continued)

d) To see if \( L \) is one to one, let \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \) with \( \mathbf{v} \neq \mathbf{w} \). Is it possible to have \( L(\mathbf{v}) = L(\mathbf{w})? \)

\[
L(\mathbf{v}) = L(\mathbf{w}) \Rightarrow L(\mathbf{v}) - L(\mathbf{w}) = 0 \Rightarrow L(\mathbf{v} - \mathbf{w}) = 0
\]

So, \( \mathbf{v} - \mathbf{w} \in \ker L \) (null space of the matrix) and

\[
\mathbf{v} - \mathbf{w} = r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}
\]

Since it is possible to have \( L(\mathbf{v}) = L(\mathbf{w}) \) when \( \mathbf{v} \neq \mathbf{w} \), \( L \) is not one to one

• Note that \( \dim \ker L + \dim \text{range } L = \dim \text{domain } L \)
Kernel and Range of a Linear Transformation

• **Theorem** - If \( L : V \rightarrow W \) is a linear transformation of an \( n \)-dimensional vector space \( V \) into a vector space \( W \), then \( \text{dim ker } L + \text{dim range } L = \text{dim } V \)

• **Proof** - Let \( k = \text{dim ker } L \). Then \( 0 \leq k \leq n \).
Consider three cases: (1) \( k = n \), (2) \( 1 \leq k < n \), and (3) \( k = 0 \)

**Case 1** - \( k = n \). Since \( \ker L \) is a subspace of \( V \) and \( \text{dim ker } L = \text{dim } V \), every basis for \( \ker L \) is a basis for \( V \). Since a vector space equals the span of its set of basis vectors, \( \ker L = V \). Now, \( L(v) = 0_W \) for all \( v \in V \). Consequently, range \( L = \{ 0_W \} \) and \( \text{dim range } L = 0 \)
Proof (continued)

**Case 2** - $1 \leq k < n$. Show that $\dim \text{range } L = n - k$.

Let $\{ v_1, v_2, \ldots, v_k \}$ be a basis for $\ker L$. This is a linearly independent set in $V$ and can be extended to a basis $S = \{ v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n \}$ for $V$.

Strategy is to show that the set of vectors

$$T = \{ L(v_{k+1}), L(v_{k+2}), \ldots, L(v_n) \}$$

is a basis for range $L$.

Specifically, need to show

a) $T$ spans range $L$

b) $T$ is linearly independent
• **Proof** (continued)

a) Let \( w \in \text{range } L \) be arbitrary. There exists a \( v \in V \) such that \( L(v) = w \). Express \( v \) in terms of the basis \( S \). \( v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \). Then

\[
L(v) = L(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n)
\]

\[
= L(\alpha_1 v_1) + L(\alpha_2 v_2) + \cdots + L(\alpha_k v_k) + L(\alpha_{k+1} v_{k+1}) + \cdots + L(\alpha_n v_n)
\]

\[
= \alpha_1 L(v_1) + \alpha_2 L(v_2) + \cdots + \alpha_k L(v_k) + \alpha_{k+1} L(v_{k+1}) + \cdots + \alpha_n L(v_n)
\]

\[
\underbrace{= 0}_{\text{since } v_1, v_2, \ldots, v_k \in \ker L}
\]

So \( w = \alpha_{k+1} L(v_{k+1}) + \alpha_{k+2} L(v_{k+2}) + \cdots + \alpha_n L(v_n) \)

and \( T \) spans \( \text{range } L \)
• **Proof** (continued)

b) To show that $T$ is linearly independent, consider

$$0_w = a_{k+1}L(v_{k+1}) + a_{k+2}L(v_{k+2}) + \cdots + a_n L(v_n)$$

$$= L(a_{k+1}v_{k+1}) + L(a_{k+2}v_{k+2}) + \cdots + L(a_n v_n)$$

$$= L(a_{k+1}v_{k+1} + a_{k+2}v_{k+2} + \cdots + a_n v_n)$$

So $a_{k+1}v_{k+1} + a_{k+2}v_{k+2} + \cdots + a_n v_n \in \ker L$ and can be written as a linear combination of $v_1, v_2, \ldots, v_k$

$$a_{k+1}v_{k+1} + \cdots + a_n v_n = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k$$

$$0 = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k - a_{k+1} v_{k+1} - \cdots - a_n v_n$$

Since S is linearly independent, the $a$ and $b$ values are all zero. So $T$ is linearly independent
Proof (continued)

Since $T$ is a basis for range $L$, $\dim \text{range } L = n - k$
So $\dim V = \dim \ker L + \dim \text{range } L$

Case 3 - $k = 0$. Since $\dim \ker L = 0$, $\ker L$ has no basis. Let $S = \{ v_1, v_2, \ldots, v_n \}$ be a basis for $V$. Let $T = \{ L(v_1), L(v_2), \ldots, L(v_n) \}$. By an argument similar to Case 2, $T$ is a basis for range $L$. So,
$\dim \text{range } L = n = \dim V$

QED
• **Corollary** - Let $L: V \rightarrow W$ and let $\dim V = \dim W$. Then
  a) If $L$ is one to one, then it is onto
  b) If $L$ is onto, then it is one to one

• **Defn** - A linear transformation $L: V \rightarrow W$ is invertible if there exists a function $L^{-1}: W \rightarrow V$ such that $L \circ L^{-1} = I_W$ and $L^{-1} \circ L = I_V$
• **Theorem** - A linear transformation $L: V \rightarrow W$ is invertible if and only if $L$ is one to one and onto. Also, $L^{-1}$ is a linear transformation and $(L^{-1})^{-1} = L$

• **Proof** - $\Rightarrow$ Let $L$ be invertible. First show that $L$ is one to one. Suppose that $L(v_1) = L(v_2)$ for some $v_1, v_2 \in V$. Then $L^{-1}(L(v_1)) = L^{-1}(L(v_2))$, implying $v_1 = v_2$. So, $L$ is one to one. Now show $L$ is onto. Let $w \in W$ be arbitrary. $L$ is invertible, so $L^{-1}$ exists. $v = L^{-1}(w) \in V$. Then $L(v) = w$ and $L$ is onto.

$\Leftarrow$ Let $L$ be one to one and onto. By function theory, the inverse function $L^{-1}$ exists.
Proof (continued)

Now show that $L^{-1}$ is a linear transformation.

a) Let $w_1, w_2 \in W$ be arbitrary. Show that

$$L^{-1}(w_1 + w_2) = L^{-1}(w_1) + L^{-1}(w_2).$$

Since $L$ is onto, there exist $v_1, v_2 \in V$ such that

$$w_1 = L(v_1) \text{ and } w_2 = L(v_2).$$

Need to show that

$$L^{-1}(w_1 + w_2) = v_1 + v_2.$$

Since $L$ is linear,

$$L(v_1+v_2) = L(v_1) + L(v_2) = w_1 + w_2.$$  

So

$$L^{-1}(w_1 + w_2) = v_1 + v_2.$$
• **Proof** (continued)
  
  b) Let \( w \in W \) and \( c \neq 0 \) be an arbitrary real. Show that \( L^{-1}(cw) = cL^{-1}(w) \). Since \( cw \in W \), there exists \( v \in V \) such that \( L(v) = cw \). Since \( L \) is linear, \( L((1/c)v) = w \). Then
  
  \[
  L^{-1}(w) = (1/c)v = (1/c)L^{-1}(cw).
  \]
  
  So \( L^{-1}(cw) = cL^{-1}(w) \).

  Thus \( L^{-1} \) is a linear transformation.

QED
Theorem - A linear transformation $L: V \rightarrow W$ is one to one if and only if the image of every linearly independent set of vectors in $V$ is a linearly independent set of vectors in $W$.

Proof - Let $S = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \}$ be a linearly independent set of vectors in $V$ and let $T = \{ L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_k) \}$.

Let $L$ be one to one. Consider

$$a_1 L(\mathbf{v}_1) + a_2 L(\mathbf{v}_2) + \cdots + a_k L(\mathbf{v}_k) = 0_W$$

Need to argue that $a_1 = a_2 = \cdots = a_k = 0$
• **Proof (continued)**

\[ a_1 L(v_1) + a_2 L(v_2) + \cdots + a_k L(v_k) = \]
\[ L(a_1 v_1) + L(a_2 v_2) + \cdots + L(a_k v_k) = \]
\[ L (a_1 v_1 + a_2 v_2 + \cdots + a_k v_k) = 0_w \]

Since \( L \) is one to one \( a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0_v \).

Since \( S \) is linearly independent,
\[ a_1 = a_2 = \cdots = a_k = 0. \] So \( T \) is linearly independent

\[ \iff \] Let the image of every set of linearly independent vectors in \( V \) be an independent set of vectors in \( W \). Let \( u, v \in V \) with \( u \neq v \). Need to show that \( L(u) \neq L(v) \). Suppose \( L(u) = L(v) \).
• **Proof (continued)**

Let \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \} be a basis for \( V \). Can express \( \mathbf{u} \) and \( \mathbf{v} \) as

\[
\mathbf{u} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \cdots + a_n \mathbf{u}_n
\]

\[
\mathbf{v} = b_1 \mathbf{u}_1 + b_2 \mathbf{u}_2 + \cdots + b_n \mathbf{u}_n
\]

\[
L(\mathbf{u}) = a_1 L(\mathbf{u}_1) + a_2 L(\mathbf{u}_2) + \cdots + a_n L(\mathbf{u}_n)
\]

\[
L(\mathbf{v}) = b_1 L(\mathbf{u}_1) + b_2 L(\mathbf{u}_2) + \cdots + b_n L(\mathbf{u}_n)
\]

\[
0_W = L(\mathbf{u}) - L(\mathbf{v})
\]

\[
= (a_1 - b_1)L(\mathbf{u}_1) + (a_2 - b_2)L(\mathbf{u}_2) + \cdots + (a_n - b_n)L(\mathbf{u}_n)
\]

By hypothesis, \( T \) is linearly independent. So \( a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n \). So, \( \mathbf{u} = \mathbf{v} \), which is a contradiction. Thus \( L(\mathbf{u}) \neq L(\mathbf{v}) \). QED
• Preliminaries
• Definition and Examples
• Kernel and Range of a Linear Transformation
• Matrix of a Linear Transformation
• Vector Spaces of Matrices and Linear Transformations
• Similarity
• Homogeneous Coordinates
Theorem - Let $L: V \rightarrow W$ be a linear transformation of an $n$-dimensional vector space $V$ into an $m$-dimensional vector space $W$ ($n \neq 0$, $m \neq 0$) and let $S = \{ v_1, v_2, \ldots, v_n \}$ be an ordered basis for $V$ and $T = \{ w_1, w_2, \ldots, w_m \}$ be an ordered basis for $W$. Then the $mxn$ matrix $A$ whose $j$th column is the coordinate vector $[L(v_j)]_T$ of $L(v_j)$ with respect to $T$ has the following property: If $y = L(x)$ for some $x \in V$, then $[y]_T = A[x]_S$. Also, $A$ is unique.
• **Proof** - Consider $L(v_j)$ for $1 \leq j \leq n$. $L(v_j) \in W$, so it can be expanded in terms of $T$

$$L(v_j) = c_{1j}w_1 + c_{2j}w_2 + \cdots + c_{mj}w_m \implies \begin{bmatrix} L(v_j) \end{bmatrix}_T = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$$

Define $A$ as the matrix whose $j$th column is $[L(v_j)]_T$.

Let $x \in V$ be arbitrary and let $y = L(x)$.

$$\begin{bmatrix} x \end{bmatrix}_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \begin{bmatrix} y \end{bmatrix}_T = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$
Linear Algebra

Linear Transformations

Matrix of a Linear Transformation

• **Proof** (continued)

\[
L(x) = L\left(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n\right)
= a_1 L(v_1) + a_2 L(v_2) + \cdots + a_n L(v_n)
\]

\[
y = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m = L(x)
\]
Proof (continued)

\[ L(x) = a_1 \left( c_{11} w_1 + c_{21} w_2 + \cdots + c_{m1} w_m \right) + \]
\[ a_2 \left( c_{12} w_1 + c_{22} w_2 + \cdots + c_{m2} w_m \right) + \]
\[ \vdots \]
\[ a_n \left( c_{1n} w_1 + c_{2n} w_2 + \cdots + c_{mn} w_m \right) \]

\[ = \left( a_1 c_{11} + a_2 c_{12} + \cdots + a_n c_{1n} \right) w_1 + \]
\[ \left( a_1 c_{21} + a_2 c_{22} + \cdots + a_n c_{2n} \right) w_2 + \]
\[ \vdots \]
\[ \left( a_1 c_{m1} + a_2 c_{m2} + \cdots + a_n c_{mn} \right) w_m \]
• **Proof** (continued)

Comparing coefficients of the $w$ vectors gives

\[ b_1 = a_1 c_{11} + a_2 c_{12} + \cdots + a_n c_{1n} \]
\[ b_2 = a_1 c_{21} + a_2 c_{22} + \cdots + a_n c_{2n} \]
\[ \vdots \]
\[ b_m = a_1 c_{m1} + a_2 c_{m2} + \cdots + a_n c_{mn} \]
• **Proof** (continued)

In matrix form

\[
\begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_m
\end{bmatrix} =
\begin{bmatrix}
    c_{11} & c_{12} & \cdots & c_{1n} \\
    c_{21} & c_{22} & \cdots & c_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{m1} & c_{m2} & \cdots & c_{mn}
\end{bmatrix}
\begin{bmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_n
\end{bmatrix}
\]

or \([y]_T = A\, [x]_S\). So the effect of \(L\) may be accomplished by letting \(A\) operate on the coordinate vector of \(x\)
• **Proof** (continued)

To show uniqueness, suppose there is a second matrix \( A^* = \begin{bmatrix} c_{ij}^* \end{bmatrix} \), which has the same properties as \( A \) but \( A^* \neq A \). Since \( A^* \neq A \) some of the elements of \( A^* \) are different from the elements of \( A \). So, suppose some elements in column \( k \) are different.

\[
[L(v_k)]_T = A [v_k]_S \quad \text{and} \quad [L(v_k)]_T = A^* [v_k]_S .
\]

So

\[
A [v_k]_S = A^* [v_k]_S
\]
• **Proof** (continued)

\[
\begin{bmatrix}
0 \\
1 \\
0 \\
\vdots \\
0 \\
0 \\
\end{bmatrix}
\]

\( [v_k]_S = \begin{bmatrix}
0 \\
1 \\
0 \\
\vdots \\
0 \\
\end{bmatrix} \leftarrow k \text{ th position} \)

\( A [v_k]_S \) is just the \( k \)th column of \( A \)

\( A^* [v_k]_S \) is just the \( k \)th column of \( A^* \)

Since \( A [v_k]_S = A^* [v_k]_S \), \( A = A^* \) and \( A \) is unique

QED
Comments

• Matrix $A$ is called the **representation** of $L$ with respect to the ordered bases $S$ and $T$

• If $L: V \rightarrow V$, can have two bases, $S$ and $T$, and get a representation of $L$ with respect to $S$ and $T$. If $S = T$, then $L$ has a representation with respect to $S$
Example

Let $L: P_2 \rightarrow P_1$ be defined by $L(p(t)) = p'(t)$ and let $S = \{ t^2, t, 1 \}$ and $T = \{ t, 1 \}$ be bases for $P_2$ and $P_1$ respectively.

a) Find the matrix $A$ associated with $L$

b) If $p(t) = 5t^2 - 3t + 2$, compute $L(p(t))$ using $A$
Example (continued)

a) Let $v_1 = t^2$, $v_2 = t$, $v_3 = 1$, $w_1 = t$, $w_2 = 1$

$L(v_1) = 2t = 2w_1$  \[\Rightarrow [L(v_1)]_T = \begin{bmatrix} 2 \\ 0 \end{bmatrix}\]

$L(v_2) = 1 = w_2$  \[\Rightarrow [L(v_2)]_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\]

$L(v_3) = 0 = 0w_1 + 0w_2$  \[\Rightarrow [L(v_3)]_T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
Example (continued)

b) \( L(p(t)) = 10t - 3 \)

\[
[p(t)]_S = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}, \quad A[p(t)]_S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}
\]

\[ \Rightarrow L(p(t)) = 10w_1 + (-3)w_2 = 10t + (-3)1 = 10t - 3 \]
Linear Algebra

Linear Transformations

Matrix of a Linear Transformation

Example

- Let \( L: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be defined as

\[
L \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

Let \( S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \)

Then \( T = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \)

\( v_1 \) \hspace{1cm} \( v_2 \) \hspace{1cm} \( v_3 \)
Linear Algebra

Linear Transformations

Matrix of a Linear Transformation

Example (continued)

\[
\begin{align*}
L\left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} L(v_1) \end{bmatrix}_T = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \\
L\left( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} L(v_2) \end{bmatrix}_T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
L\left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} L(v_3) \end{bmatrix}_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{align*}
\]

\[
A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}
\]
Linear Algebra

Linear Transformations

Topics

• Preliminaries
• Definition and Examples
• Kernel and Range of a Linear Transformation
• Matrix of a Linear Transformation
• Vector Spaces of Matrices and Linear Transformations
• Similarity
• Homogeneous Coordinates
Comments

• Have shown that the set of $m \times n$ matrices $\mathbb{R}^n_m$ is a vector space

• Want to show that the set $U$ of all linear transformations from $V$ to $W$ forms a vector space

• Need to define the operations for the vector space
  a) sum of two linear transformations
  b) scalar times a linear transformation
• **Defn** - Let $L_1: V \rightarrow W$ and $L_2: V \rightarrow W$. Define the sum of $L_1$ and $L_2$, $L = L_1 \oplus L_2$ as follows

$$L(x) = L_1(x) + L_2(x) \quad \forall x \in V$$

Note: This is just the definition of the sum of two functions

• **Defn** - Let $L: V \rightarrow W$ be a linear transformation and let $c$ be real. Define the scalar multiple of $c$ and $L$, $c \circ L$, as

$$(c \circ L)(x) = cL(x) \quad \forall x \in V$$
• Verification of vector space properties for U, the set of all linear transformations from V to W with the operations $\oplus$ and $\circ$, is straightforward except for

a) zero vector - define $0(x) = 0_w \quad \forall x \in V$

b) additive inverse - Let $L \in U$, define $-L$ as $(-1) \circ L$

• Since U is a vector space, what is its dimension? Answer question via a basis
• **Defn** - Let \( S = \{ L_1, L_2, \ldots, L_k \} \) be a set of linear transformations. \( S \) is **linearly dependent** if there exist scalars \( a_1, a_2, \ldots, a_k \), not all zero, such that

\[
(a_1 \circ L_1) \oplus (a_2 \circ L_2) \oplus \cdots \oplus (a_k \circ L_k) = 0
\]

where \( 0 \) is the zero linear transformation.
Example

- Consider linear transformations $L_1$, $L_2$, $L_3$ from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined as

$$L_1([x_1, x_2]) = [x_1 + x_2, 2x_1, x_2]$$
$$L_2([x_1, x_2]) = [x_2 - x_1, 2x_1 + x_2, x_1]$$
$$L_3([x_1, x_2]) = [3x_1, -2x_2, x_1 + 2x_2]$$

Determine if $S = \{ L_1, L_2, L_3 \}$ is linearly dependent

Suppose $\left( a_1 \circ L_1 \right) \oplus \left( a_2 \circ L_2 \right) \oplus \left( a_3 \circ L_3 \right) = 0$ where $a_1, a_2, a_3$ are real. This equation means

$$a_1 L_1(x) + a_2 L_2(x) + a_3 L_3(x) = 0_{\mathbb{R}^3} \quad \forall x \in \mathbb{R}^2, x = [x_1, x_2]$$
Example (continued)

\[ \mathbf{0}_{\mathbb{R}^3} = \begin{bmatrix} 0, 0, 0 \end{bmatrix} \]

\[ = a_1 L_1(\mathbf{x}) + a_2 L_2(\mathbf{x}) + a_3 L_3(\mathbf{x}) \]

\[ = a_1 \begin{bmatrix} x_1 + x_2, 2x_1, x_2 \end{bmatrix} + a_2 \begin{bmatrix} x_2 - x_1, 2x_1 + x_2, x_1 \end{bmatrix} + \]

\[ a_3 \begin{bmatrix} 3x_1, -2x_2, x_1 + 2x_2 \end{bmatrix} \]

\[ = \begin{bmatrix} a_1(x_1 + x_2) + a_2(x_2 - x_1) + a_3(3x_1), \\
                  a_1(2x_1) + a_2(2x_1 + x_2) + a_3(-2x_2), \\
                  a_1x_2 + a_2x_1 + a_3(x_1 + 2x_2) \end{bmatrix} \]
Example (continued)

\[ a_1(x_1 + x_2) + a_2(x_2 - x_1) + 3a_3x_1 = 0 \]
\[ 2a_1x_1 + a_2(2x_1 + x_2) - 2a_3x_2 = 0 \]
\[ a_1x_2 + a_2x_1 + a_3(x_1 + 2x_2) = 0 \]

This must be true \( \forall x_1, x_2 \). Pick particular values \( x_1 = 1, x_2 = 0 \)

\[ a_1 - a_2 + 3a_3 = 0 \]
\[ 2a_1 + 2a_2 = 0 \]
\[ a_2 + a_3 = 0 \]

The only solution is \( a_1 = 0, a_2 = 0, a_3 = 0 \). So \( S = \{ L_1, L_2, L_3 \} \) is linearly independent
• **Theorem** - Let $U$ be the set of all linear transformations of $V$ into $W$ where $\dim V = n$ and $\dim W = m$, $n \neq 0$, $m \neq 0$, and operations in $U$ are $\oplus$ and $\circ$. $U$ is isomorphic to the vector space $m \times n$ of all $m \times n$ matrices.

• **Proof** - Strategy is to pick a basis for $V$ and for $W$ and map $L$ to its matrix representation with respect to these bases. This gives a mapping from $U$ to $m \times n$. Need to show that the mapping
1) is one to one
2) is onto
3) preserves vector operations
• **Proof (continued)**

Let $S = \{ v_1, v_2, \ldots, v_n \}$ be a basis for $V$ and let $T = \{ w_1, w_2, \ldots, w_m \}$ be a basis for $W$. Define a mapping $M: U \rightarrow m \times n \mathbb{R}$ as $M(L) =$ matrix representing $L$ with respect to $S$ and $T$.

1) Show $M$ is one to one. Let $L_1, L_2 \in U$ with $L_1 \neq L_2$. Need to show $M(L_1) \neq M(L_2)$. Since $L_1 \neq L_2$, $\exists \ v \in V$ such that $L_1(v) \neq L_2(v)$. $v$ can be expressed as a linear combination of elements of $S$. So, must have $L_1(v_j) \neq L_2(v_j)$ for some $1 \leq j \leq n$. The $j$th column of $M(L_1)$ is $[L_1(v_j)]_T$. The $j$th column of $M(L_2)$ is $[L_2(v_j)]_T$. Since $L_1(v_j) \neq L_2(v_j)$, $[L_1(v_j)]_T \neq [L_2(v_j)]_T$. So $M(L_1) \neq M(L_2)$. 
2) Show $M$ is onto. Let $A = [a_{ij}]$ be an arbitrary $m \times n$ matrix. Define a linear transformation $L: V \rightarrow W$ by $L(v_i) = \sum_{k=1}^{m} a_{ki} w_k$ for $1 \leq i \leq n$

Note: it is sufficient to define $L$ on $S$ since for any $x \in V, x = c_1 v_1 + \cdots + c_n v_n$, then

$$L(x) = \sum_{i=1}^{n} c_i L(v_i)$$

$L$ is a linear transformation and the matrix of $L$ with respect to bases $S$ and $T$ is $A$. So $M$ is onto.
• **Proof** (continued)

3) Show that $M$ preserves vector addition and scalar multiplication. Let $L_1, L_2 \in U$ be arbitrary. Let $M(L_1) = A = [a_{ij}]$ and $M(L_2) = B = [b_{ij}]$. First show that $M(L_1 \oplus L_2) = A + B$. The $j$th column of $M(L_1 \oplus L_2)$ is

$$
\left[ (L_1 \oplus L_2)(v_j) \right]_T = \left[ L_1(v_j) + L_2(v_j) \right]_T 
$$

$$
= \left[ L_1(v_j) \right]_T + \left[ L_2(v_j) \right]_T 
$$

So $j$th column of $M(L_1 \oplus L_2)$ is sum of $j$th columns of $M(L_1) = A$ and $M(L_2) = B$. So $M(L_1 \oplus L_2) = A + B$
• **Proof** (continued)

Consider scalar multiplication. Let $M(L) = A$ and real $c$ be arbitrary. The $j$th column of $M(c \circ L)$ is

$$
\left[ (c \circ L)(v_j) \right]_T = \left[ cL(v_j) \right]_T = c \left[ L(v_j) \right]_T
$$

So $M(c \circ L) = cA$

$\therefore$ $U$ and $mR_n$ are isomorphic

QED
• Corollary - \( \text{dim } U = mn \)

• Since linear transformations are just functions, can form composition of those functions. Following theorem shows that matrix of composition is simply related to matrices of individual transformations
• **Theorem** - Let $V_1$, $V_2$, $V_3$ be vector spaces with $\dim V_1 = n$, $\dim V_2 = m$, $\dim V_3 = p$. Let $L_1 : V_1 \rightarrow V_2$, $L_2 : V_2 \rightarrow V_3$ be linear transformations. Let $P$, $S$, $T$ be bases for $V_1$, $V_2$, $V_3$ respectively. Then

$$M(L_2 \circ L_1) = M(L_2)M(L_1)$$

where $\circ$ is the composition of functions.
• **Proof** - Let $M(L_1) = A$, with respect to bases $P$ and $S$. Let $M(L_2) = B$, with respect to bases $S$ and $T$. Let $x \in V_1$ be arbitrary. Then $[L_1(x)]_S = A [x]_P$. For any $y \in V_2$, $[L_2(y)]_T = B [y]_S$

$$
\left[ (L_2 \circ L_1)(x) \right]_T = \left[ L_2 \left( L_1(x) \right) \right]_T
$$

$$
= B \left[ L_1(x) \right]_S = B \left( A [x]_P \right) = (BA) [x]_P
$$

Have proved that matrix of a linear transformation with respect to a particular basis is unique. So

$$
M(L_2 \circ L_1) = AB = M(L_2)M(L_1)
$$

QED
Preliminaries

Definition and Examples

Kernel and Range of a Linear Transformation

Matrix of a Linear Transformation

Vector Spaces of Matrices and Linear Transformations

Similarity

Homogeneous Coordinates
Linear Algebra

Linear Transformations

Similarity

Comments

• Let $L: V \rightarrow W$ be a linear transformation, where $\dim V = n \neq 0$ and $\dim W = m \neq 0$ and the spaces have ordered bases $S = \{ v_1, v_2, \ldots, v_n \}$ for $V$ and $T = \{ w_1, w_2, \ldots, w_m \}$ for $W$. Have seen how to construct a matrix $A$ that represents $L$ with respect to these bases. Specifically, the $j$th column of $A$ is $[L(v_j)]_T$

• Also know that picking a different basis in either $V$ or $W$ gives a different matrix

• Since all of these matrices represent $L$, they ought to be related, i.e. we ought to be able to get one matrix from another
• **Theorem** - Let $L: V \rightarrow W$ be a linear transformation, where $\dim V = n \neq 0$ and $\dim W = m \neq 0$. Let $S = \{v_1, v_2, \ldots, v_n\}$ and $S' = \{v'_1, v'_2, \ldots, v'_n\}$ be ordered bases for $V$ with transition matrix $P$ from $S'$ to $S$. Let $T = \{w_1, w_2, \ldots, w_m\}$ and $T' = \{w'_1, w'_2, \ldots, w'_m\}$ be ordered bases for $W$ with transition matrix $Q$ from $T'$ to $T$. If $A$ is the representation of $L$ with respect to $S$ and $T$, then $Q^{-1}AP$ is the representation of $L$ with respect to $S'$ and $T'$.
• **Proof** - Recall the definition of the transition matrices:

\[
\begin{align*}
[x]_S &= P[x]_S', \quad \forall x \in V \\
[y]_T &= Q[y]_T', \quad \forall y \in W
\end{align*}
\]

- $j$th column of $P$ is coordinate vector $[v'_j]_S$ of $v'_j$ with respect to $S$
- $j$th column of $Q$ is coordinate vector $[w'_j]_T$ of $w'_j$ with respect to $T$
• **Proof** (continued)

Let $A$ be the representation of $L$ with respect to $S$ and $T$, then $[L(x)]_T = A[x]_S$.

Also,

\[
[L(x)]_T = Q[L(x)]_{T'}, \quad [x]_S = P[x]_{S'}
\]

\[
\Rightarrow Q[L(x)]_{T'} = AP[x]_{S'} \Rightarrow [L(x)]_{T'} = Q^{-1}AP[x]_{S'}
\]

So $Q^{-1}AP$ is the representation of $L$ with respect to $S'$ and $T'$

**QED**
Comments

- Let $\mathbf{B} = \mathbf{Q}^{-1} \mathbf{A} \mathbf{P}$. Can calculate the effect of $\mathbf{L}$ on $\mathbf{x} \in \mathbf{V}$ two ways in terms of $\mathbf{S}'$ and $\mathbf{T}'$.

- Note that $\mathbf{A}$ and $\mathbf{B}$ are equivalent matrices.
Example

• Define $L: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by
  
  \[ L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 \\ x_2 - x_3 \end{pmatrix} \]

Bases for $\mathbb{R}^3$ $S = \{v_1, v_2, v_3\}$

\[
S = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
v_1 \\
v_2 \\
v_3 \\
\end{bmatrix}
\]

Bases for $\mathbb{R}^2$ $T = \{w_1, w_2\}$

\[
T = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

\[
w_1 \\
w_2 \\
\end{bmatrix}
\]

\[
S' = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
\end{bmatrix}
\]

\[
v_1' \\
v_2' \\
v_3' \\
\end{bmatrix}
\]

\[
T' = \begin{bmatrix}
1 & 1 \\
1 & 3 \\
\end{bmatrix}
\]

\[
w_1' \\
w_2' \\
\end{bmatrix}
\]
Linear Algebra

Linear Transformations

Example (continued)

\[
A = \begin{bmatrix} [L(v_1)]_T & [L(v_2)]_T & [L(v_3)]_T \end{bmatrix}
\]

\[
L(v_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow [L(v_1)]_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{Since } T \text{ is the natural basis}
\]

\[
L(v_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow [L(v_2)]_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{Since } T \text{ is the natural basis}
\]

\[
L(v_3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow [L(v_3)]_T = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{Since } T \text{ is the natural basis}
\]

\[
A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}
\]
Example (continued)

Calculate representation of $L$ with respect to $S'$ and $T'$ two ways

Calculate $P$ - Columns of $P$ are $\begin{bmatrix} \mathbf{v}'_1 \end{bmatrix}_S$, $\begin{bmatrix} \mathbf{v}'_2 \end{bmatrix}_S$, $\begin{bmatrix} \mathbf{v}'_3 \end{bmatrix}_S$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Since $S$ is a natural basis $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Calculate $Q$ - Columns of $Q$ are $\begin{bmatrix} \mathbf{w}'_1 \end{bmatrix}_T$, $\begin{bmatrix} \mathbf{w}'_2 \end{bmatrix}_T$

$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

Since $T$ is a natural basis $Q = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$
Example (continued)

\[ Q^{-1} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \]

\[ B = Q^{-1}AP = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} 1 & 3/2 & 2 \\ 0 & -1/2 & -1 \end{bmatrix} \]
Example (continued)

Compute $\mathbf{B}$ directly

Columns of $\mathbf{B}$ are $\left[ L(v'_1) \right]_{T'}$, $\left[ L(v'_2) \right]_{T'}$, $\left[ L(v'_3) \right]_{T'}$.

$L(v'_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  \hspace{1cm} $\left[ L(v'_1) \right]_{T'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$L(v'_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  \hspace{1cm} $\left[ L(v'_2) \right]_{T'} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$

$L(v'_3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  \hspace{1cm} $\left[ L(v'_3) \right]_{T'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} 1 & 3/2 & 2 \\ 0 & -1/2 & -1 \end{bmatrix}$
• **Corollary** - Let $L: V \rightarrow V$ be a linear operator on an $n$-dimensional vector space $V$. Let $S = \{ v_1, v_2, \ldots, v_n \}$ and $S' = \{ v'_1, v'_2, \ldots, v'_n \}$ be ordered bases for $V$, with $P$ being the transition matrix from $S'$ to $S$. If $A$ is the representation of $L$ with respect to $S$, then $P^{-1}AP$ is the representation of $L$ with respect to $S'$.
• **Defn** - The rank of $L: V \rightarrow W$, notation $\text{rank } L$, is the rank of any matrix representing $L$

• Note: $\text{rank } L$ is well defined since any two matrices representing $L$ are equivalent and thus have the same rank
• **Theorem** - Let $L: V \rightarrow W$ be a linear transformation. Then $\text{rank } L = \text{dim range } L$

• **Proof** - Let $\text{dim } V = n$, $\text{dim } W = m$ and $\text{dim range } L = r$. We have proved a theorem that says $\text{dim ker } L + \text{dim range } L = n$. Then $\text{dim ker } L = n - r$. Let $\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \ldots, \mathbf{v}_n$ be a basis for ker $L$. This can be extended to a basis $\mathbf{S} = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n \}$ for $V$.

The vectors $\mathbf{w}_1 = L(\mathbf{v}_1), \mathbf{w}_2 = L(\mathbf{v}_2), \ldots, \mathbf{w}_r = L(\mathbf{v}_r)$ span range $L$. Since there are $r = \text{dim range } L$ of them, they form a basis for range $L$. This can be extended to a basis $\mathbf{T} = \{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{w}_{r+1}, \ldots, \mathbf{w}_m \}$ for $W$. 
• **Proof** (continued)

Now let \( A \) be the matrix that represents \( L \) with respect to \( S \) and \( T \). The columns of \( A \) are

\[
\begin{align*}
[L(v_i)]_T &= [w_i] = e_i & i = 1, 2, \ldots, r \\
[L(v_i)]_T &= [0_w]_T = 0_{R^m} & i = r + 1, r + 2, \ldots, n 
\end{align*}
\]

So \( A = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \)

Thus \( \text{rank } L = \text{rank } A = r = \text{dim range } L \)

QED
• **Defn** - If \( A \) and \( B \) are are \( n \times n \) matrices, then \( B \) is similar to \( A \) if there is a nonsingular matrix \( P \) such that \( B = P^{-1}AP \)
**Theorem** - Let V be any n-dimensional vector space and let A and B be any n x n matrices. Then A and B are similar if and only if A and B represent the same linear transformation L: V → V with respect to two ordered bases for V.

**Proof** - Let A and B be similar. Then there is a nonsingular matrix \( P = [ p_{ij} ] \) such that \( B = P^{-1}AP \). Let \( S = \{ v_1, v_2, \ldots, v_n \} \) be an ordered basis for V and define a linear transformation on V by \( [L(x)]_S = A[x]_S \) for all \( x \) in V.

Now define a new basis for V by taking appropriate linear combinations of vectors in S.
• **Proof (continued)** -

Define a set of vectors \( T = \{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \} \) as

\[
\mathbf{w}_j = \sum_{i=1}^{n} p_{ij} \mathbf{v}_i, \quad 1 \leq j \leq n,
\]

and show that \( T \) is a basis for \( V \) by showing that it is linearly independent and appealing to an earlier theorem that says that a set of \( n \) linearly independent vectors in an \( n \)-dimensional space is a basis.

Consider

\[
0 = a_1 \mathbf{w}_1 + a_2 \mathbf{w}_2 + \cdots + a_n \mathbf{w}_n
\]

\[
= a_1 \sum_{i=1}^{n} p_{i1} \mathbf{v}_i + a_2 \sum_{i=1}^{n} p_{i2} \mathbf{v}_i + \cdots + a_n \sum_{i=1}^{n} p_{in} \mathbf{v}_i
\]

\[
= \left( \sum_{j=1}^{n} p_{1j} a_j \right) \mathbf{v}_1 + \left( \sum_{j=1}^{n} p_{2j} a_j \right) \mathbf{v}_2 + \cdots + \left( \sum_{j=1}^{n} p_{nj} a_j \right) \mathbf{v}_n
\]
Proof (continued) -
Since S is linearly independent \((\sum_{j=1}^{n} p_{ij}a_j) = 0, \quad 1 \leq i \leq n\)
or equivalently \(Pa = 0\), where \(a = [a_1 \ a_2 \ \ldots \ a_n]^T\). Since \(P\) is nonsingular, the only solution is \(a = 0\).
Thus \(T\) is linearly independent and is a basis for \(V\).
The definition of \(T\), \(w_j = \sum_{i=1}^{n} p_{ij}v_i, \quad 1 \leq j \leq n\), implies that \(P\) is the transition matrix from \(T\) to \(S\), i.e. \([y]_S = P[y]_T\). Then, recalling the definition of \(L\),
\[
\begin{align*}
\left[ L(x) \right]_S &= P \left[ L(x) \right]_T \rightarrow A \left[ x \right]_S = P \left[ L(x) \right]_T \rightarrow \\
\left[ L(x) \right]_T &= P^{-1} A \left[ x \right]_S = P^{-1} AP \left[ x \right]_T
\end{align*}
\]
So, the matrix of \(L\) with respect to \(T\) is \(P^{-1}AP = B\)
Proof (continued) -

\[\iff\] By the preceding corollary, any two matrix representations of a linear transformation are similar.

QED
• Theorem - If $A$ and $B$ are similar $n \times n$ matrices, then $\text{rank } A = \text{rank } B$.

• Proof - By the preceding theorem, $A$ and $B$ represent the same linear transformation $L: \mathbb{R}^n \to \mathbb{R}^n$ with respect to different bases. Since the rank of $L$ is defined uniquely as the rank of any matrix representing it, $\text{rank } A = \text{rank } L = \text{rank } B$.

QED
Linear Algebra

Linear Transformations

Topics

• Preliminaries
• Definition and Examples
• Kernel and Range of a Linear Transformation
• Matrix of a Linear Transformation
• Vector Spaces of Matrices and Linear Transformations
• Similarity
• Homogeneous Coordinates
A commonly used technique in computer graphics is the homogeneous coordinate transformation, which combines a sequence of translations, scalings and rotations into a single matrix which is then applied to the vertices of a geometric object.

- This allows a compact representation of the combined operations that is easy to apply.
- Also, the individual transformations can be implemented in hardware in a high-end workstation to permit the rotation of an object on the screen by means of turning a knob.
Motivating Example

- Rotate the cube about an axis parallel to the $z$ axis passing through the point $(1, 2, 3)$, by angles of $\Delta \theta$, $2 \Delta \theta$, $3 \Delta \theta$, etc. from its original position (i.e. successive rotations by angles of $\Delta \theta$). After each rotation, display the rotated cube to give the visual effect of a spinning cube.

Vertices at $(1 \pm 1, 2 \pm 1, 3 \pm 1)$. Cube's faces are parallel to coordinate planes.
Basic Coordinate Operations

- The application of any of these operations to a cube is accomplished by applying the operation to each vertex of the cube.

- **Translation**: The translation of \((x, y, z)\) by the translation vector \((t_x, t_y, t_z)\) yields the point whose coordinates are \((x + t_x, y + t_y, z + t_z)\), i.e. \((x, y, z)\) is moved to \((x + t_x, y + t_y, z + t_z)\).

- **Scaling**: The scaling of \((x, y, z)\) by the scaling vector \((s_x, s_y, s_z)\), with \(s_x > 0, s_y > 0\) and \(s_z > 0\), yields the point with coordinates \((xs_x, ys_y, zs_z)\), i.e. the point's coordinates are scaled by these amounts.
Basic Coordinate Operations

- **Rotation**: Simple rotations are done about the $x$-axis, $y$-axis and $z$-axis.
  - $x$-axis: If $(x, y, z)$ is rotated about the $x$-axis by an angle $\theta$ to a new point $(x', y', z')$, the coordinates are related by
    
    \[
    \begin{align*}
    x' &= x \\
y' &= y \cos \theta - z \sin \theta \\
z' &= y \sin \theta + z \cos \theta
    \end{align*}
    \]
  
  - $y$-axis: If $(x, y, z)$ is rotated about the $y$-axis by an angle $\theta$ to a new point $(x', y', z')$, the coordinates are related by
    
    \[
    \begin{align*}
    x' &= z \sin \theta + x \cos \theta \\
y' &= y \\
z' &= z \cos \theta - x \sin \theta
    \end{align*}
    \]
Basic Coordinate Operations

- $z$-axis: If $(x, y, z)$ is rotated about the $x$-axis by an angle $\theta$ to a new point $(x', y', z')$, the coordinates are related by

$$
x' = x \cos \theta - y \sin \theta
$$

$$
y' = x \sin \theta + y \cos \theta
$$

$$
z' = z
$$
In the motivating example, rotation of the cube by $\Delta \theta$ about a line through $(1, 2, 3)$ parallel to the $z$-axis can be expressed in terms of the coordinate operations defined on the previous slides:

1. Translate each vertex by $(-1, -2, -3)$ to place the center of the cube at the origin and cause the axis of rotation to coincide with the $z$-axis.
2. Rotate the cube about the $z$-axis by an angle of $\Delta \theta$.
3. Translate the rotated cube by $(1, 2, 3)$ to put it back in position.

The three steps above will perform the rotation and successive applications of the process will perform subsequent rotations.
Matrix Procedure

- Represent the point \((x, y, z)\) as a 4 x 1 matrix \(X = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\).

- **Translation**: Translation by \((t_x, t_y, t_z)\) can be accomplished as

\[
\begin{bmatrix}
1 & 0 & 0 & t_x \\
0 & 1 & 0 & t_y \\
0 & 0 & 1 & t_z \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1 \\
\end{bmatrix} =
\begin{bmatrix}
x + t_x \\
y + t_y \\
z + t_z \\
1 \\
\end{bmatrix}
\Rightarrow T(t_x, t_y, t_z)X = X'
\]
Matrix Procedure

• **Scaling**: Scaling by \((s_x, s_y, s_z)\) can be accomplished as

\[
\begin{bmatrix}
  s_x & 0 & 0 & 0 \\
  0 & s_y & 0 & 0 \\
  0 & 0 & s_z & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
= \begin{bmatrix}
  xs_x \\
  ys_y \\
  zs_z \\
  1
\end{bmatrix}
\Rightarrow S(s_x, s_y, s_z)X = X'
\]
Matrix Procedure

- **Rotation**: Rotation about the $x$-axis by $\theta$ can be accomplished as

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 & 0 \\
0 & \sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
= \begin{bmatrix}
x \\
y \cos \theta - z \sin \theta \\
y \sin \theta + z \cos \theta \\
1
\end{bmatrix}$$

$$\Rightarrow R_x(\theta)X = X'$$
Matrix Procedure

- **Rotation**: Rotation about the y-axis by $\theta$ can be accomplished as

\[
\begin{bmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
= 
\begin{bmatrix}
x \cos \theta + z \sin \theta \\
y \\
-x \sin \theta + z \cos \theta \\
1
\end{bmatrix}
\]

$$\Rightarrow R_y(\theta)X = X'$$
Matrix Procedure

- **Rotation**: Rotation about the $z$-axis by $\theta$ can be accomplished as

\[
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix} =
\begin{bmatrix}
x \cos \theta - y \sin \theta \\
x \sin \theta + y \cos \theta \\
z \\
1
\end{bmatrix}
\]

$\Rightarrow R_z(\theta)X = X'$
Matrix Procedure

• Inverses of the matrices are easy to compute

\[ T^{-1}(t_x, t_y, t_z) = T(-t_x, -t_y, -t_z) \]

\[ S^{-1}(s_x, s_y, s_z) = S(1/s_x, 1/s_y, 1/s_z) \]

\[ R_x^{-1}(\theta) = R_x(-\theta) \]

\[ R_y^{-1}(\theta) = R_y(-\theta) \]

\[ R_z^{-1}(\theta) = R_z(-\theta) \]
Matrix Procedure

- Note that any sequence of coordinate operations may be performed by multiplying by the appropriate matrices.
- The sequence of operations may be inverted by multiplying by the inverse matrices in reverse order.
Motivating Example (continued)

- The operations in the example can be accomplished as

  1. Translate by \((-1, -2, -3)\) → \(T(-1, -2, -3)\)
  2. Rotate about the \(z\)-axis by \(\Delta \theta\) → \(R_z(\Delta \theta)\)
  3. Translate by \((1, 2, 3)\) → \(T(1, 2, 3)\)
Motivating Example (continued)

- Define $M(\Delta \theta)$ as

\[
M(\Delta \theta) = T(1,2,3)R_z(\Delta \theta)T(-1,-2,-3)
\]
Linear Algebra

Linear Transformations

Motivating Example (continued)

\[ M^2(\Delta \theta) \]
\[ = T(1, 2, 3) R_z(\Delta \theta) T(-1, -2, -3) T(1, 2, 3) R_z(\Delta \theta) T(-1, -2, -3) \]
\[ = T(1, 2, 3) R_z(\Delta \theta) R_z(\Delta \theta) T(-1, -2, -3) \]
\[ = T(1, 2, 3) R_z^2(\Delta \theta) T(-1, -2, -3) \]
\[ = T(1, 2, 3) R_z(2\Delta \theta) T(-1, -2, -3) \]

By an inductive argument, can show

\[ M^n(\Delta \theta) = T(1, 2, 3) R_z(n\Delta \theta) T(-1, -2, -3) = M(n\Delta \theta) \]
Another Example

- Consider problem of rotating the cube by $\Delta \theta$ about an axis passing through the vertices $(0, 1, 2)$ and $(2, 3, 4)$.
Another Example (continued)

- Other than simplifying the discussion, there is nothing special about the points (0, 1, 2) and (2, 3, 4) or the fact that they are vertices of the cube. One could just as readily talk about rotation about an axis through the points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\).

- It does make a difference whether one considers the axis of rotation as going from (0, 1, 2) to (2, 3, 4) or from (2, 3, 4) to (0, 1, 2). The second choice reverses the sense of the rotation from the first choice.
Another Example (continued)

- Procedure is
  1. Translate the cube by the translation vector \((-1, -2, -3)\) This places the points which determine the rotation axis at \((-1, -1, -1)\) and \((1, 1, 1)\)
  2. Rotate the axis of rotation into the \(z\)-axis by the following steps
     a. Rotate by \(\pi/4\) about the \(z\)-axis to put \((1, 1, 1)\) and \((-1, -1, -1)\) in the \(yz\)-plane
Another Example (continued)

(b) Rotate by $\arctan(\sqrt{2})$ about the $x$-axis to put $(1, 1, 1)$ and $((-1,-1,-1)$ onto the $z$-axis

(3) Rotate about the $z$ axis by $\Delta \theta$

(4) Undo steps 1 and 2

• The matrix $M(\Delta \theta)$ to do this is

$$M(\Delta \theta) = \begin{align*}
&\mathbf{T}^{-1}(-1,-2,-3)\mathbf{R}_z^{-1}(\pi/4)\mathbf{R}_x^{-1}(\arctan \sqrt{2})\mathbf{R}_z(\Delta \theta)\mathbf{R}_x(\arctan \sqrt{2})\mathbf{R}_z(\pi/4)\mathbf{T}(-1,-2,-3) \\
=&\mathbf{T}(1,2,3)\mathbf{R}_z(-\pi/4)\mathbf{R}_x(-\arctan \sqrt{2})\mathbf{R}_z(\Delta \theta)\mathbf{R}_x(\arctan \sqrt{2})\mathbf{R}_z(\pi/4)\mathbf{T}(-1,-2,-3)
\end{align*}$$