Linear Algebra

 Determinants

• Preliminaries
• Definition
• Properties of Determinants
• Cofactor Expansion
• Inverse of a Matrix
• Other Applications of Determinants
• Computing Determinants
Linear Algebra

Determinants

Topics

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• Consider the system of linear equations

\[ \begin{align*}
ax_1 + bx_2 &= y_1 \\
cx_1 + dx_2 &= y_2
\end{align*} \]

which can be written in matrix form as

\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} =
\begin{bmatrix}
    y_1 \\
    y_2
\end{bmatrix}
\]

Can show that this system has a solution if and only if \( ad - bc \neq 0 \). The number \( ad - bc \), which also appears in the solution of the above equations, is called the determinant of the coefficient matrix.

• Want to generalize this to a square matrix of any size.
Uses of Determinants

- Condition for singularity is \( \det (A) = 0 \).
  
  Eigenvalues of \( A \) are those values \( \lambda \) that make the matrix \( A - \lambda I \) singular. Condition \( \det (A - \lambda I) = 0 \) gives a polynomial whose roots are the eigenvalues.

- Explicit formulas for solutions of long-standing problems:
  
  - solution of linear systems
  
  - computation of matrix inverse

  - Explicit formula for \( A^{-1} \) is of limited computational utility, but it does show how \( A^{-1} \) depends on the \( n^2 \) entries of \( A \) and how it varies when those entries vary.
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• Computing Determinants
• **Defn** - Let $S = \{ 1, 2, \ldots, n \}$ be the integers 1 through $n$, arranged in ascending order. A rearrangement $j_1j_2 \ldots j_n$ of the elements of $S$ is called a permutation of $S$. A permutation of $S$ is a one to one mapping of $S$ onto itself.

• The number of permutations of $S = \{ 1, 2, \ldots, n \}$ is $n!$.

• The set of all permutations of the elements of $S$ is denoted by $S_n$.

• It will be helpful to classify permutations as even or odd.
• **Defn** - A permutation \( j_1 j_2 \ldots j_n \) of S is said to have an **inversion** if a larger integer, \( j_r \), precedes a smaller one, \( j_s \).

• **Example** - Let \( S = \{ 1, 2, 3 \} \). Then \( S_3 = \{ 123, 132, 213, 231, 312, 321 \} \)

  123 - 0 inversions
  132 - 1 inversion
  213 - 1 inversion
  231 - 2 inversions
  312 - 2 inversions
  321 - 3 inversions
• **Defn** - A permutation is called **even** if the total number of inversions is even

• **Defn** - A permutation is called **odd** if the total number of inversions is odd

• **Example** - 4312 contains 5 inversions and is an odd permutation. 1423 contains 2 inversions and is an even permutation
• **Defn** - Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **determinant** of $A$ is defined as

$$
\det(A) = \sum (\pm) a_{j_1} a_{j_2} \cdots a_{j_n}
$$

where the summation is over all permutations $j_1 j_2 \cdots j_n$ of $S = \{1, 2, \ldots, n\}$. The sign is $+$ if $j_1 j_2 \cdots j_n$ is an even permutation and is $-$ if $j_1 j_2 \cdots j_n$ is an odd permutation.
Comments

- det (A) is a function from the set of $n \times n$ matrices into the real numbers.
- Each term $\pm a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}$ of det (A) has row subscripts in natural order and column subscripts in the order $j_1 j_2 \cdots j_n$. Thus each term is a product of $n$ elements of A, with exactly one entry from each row of A and exactly one entry from each column of A.
- det (A) is also written $|A|$. 
Examples

• Let $A = \begin{bmatrix} a_{11} \end{bmatrix}$, a $1 \times 1$ matrix, then $\det(A) = a_{11}$

• Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$
Linear Algebra

Determinants

Examples

- Let \( A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix} \)

\[
\text{det} (A) = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} \\
- a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}
\]
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**Theorem** - Let \( A \) be a square matrix. Then 
\[
\det (A) = \det (A^T)
\]

**Proof** - Let \( A = [a_{ij}] \) and \( A^T = [b_{ij}] \) where \( b_{ij} = a_{ji} \)
\[
\det (A^T) = \sum (\pm) b_{1j_1} b_{2j_2} \cdots b_{nj_n} = \sum (\pm) a_{j_1} a_{j_2} \cdots a_{j_n}
\]
Consider a term of the sum
\[
b_{1j_1} b_{2j_2} \cdots b_{nj_n} = a_{j_1} a_{j_2} \cdots a_{j_n} = a_{1k_1} a_{2k_2} \cdots a_{nk_n}
\]
where \( k_1 k_2 \ldots k_n \) is a permutation of \( 1, 2, \ldots, n \). So, 
\[
det (A) \text{ and } \det (A^T) \text{ contain the same terms with possibly a difference in sign. Can argue that } k_1 k_2 \ldots k_n \text{ is the inverse of } j_1 j_2 \ldots j_n, \text{ thus both have the same parity. So, the signs of corresponding terms in } \det (A) \text{ and } \det (A^T) \text{ are the same and } \det (A) = \det (A^T).
\]
\[QED\]
• Comment - Since $\det(A) = \det(A^T)$, the properties of determinants with respect to row manipulations are true also for the corresponding column manipulations.
• **Theorem** - If matrix \( \mathbf{B} \) is obtained from matrix \( \mathbf{A} \) by interchanging two rows (columns) of \( \mathbf{A} \), then \( \det ( \mathbf{B} ) = - \det ( \mathbf{A} ) \)

• **Proof** - Suppose \( \mathbf{B} \) is obtained from \( \mathbf{A} \) by interchanging rows \( r \) and \( s \) of \( \mathbf{A} \). Without loss of generality, let \( r < s \). Then \( b_{rj} = a_{sj}, b_{sj} = a_{rj} \), for all \( j \) and \( b_{ij} = a_{ij} \) for all \( j \) if \( i \neq r, i \neq s \).

\[
\det ( \mathbf{B} ) = \sum (\pm) b_{1j_1} b_{2j_2} \cdots b_{rj_r} \cdots b_{sj_s} \cdots b_{nj_n}
\]

\[
= \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{sj_s} \cdots a_{rj_r} \cdots a_{nj_n}
\]

\[
= \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{rj_r} \cdots a_{sj_s} \cdots a_{nj_n}
\]
Proof (continued)

The permutation $j_1j_2 \ldots j_s \ldots j_r \ldots j_n$ results from the permutation $j_1j_2 \ldots j_r \ldots j_s \ldots j_n$ by an interchange of two numbers. Can show that the number of inversions in the first permutation differs from the number of inversions in the second permutation by an odd number. So each term in $\det(B)$ has the opposite sign of the corresponding term in $\det(A)$. Therefore

$$\det(B) = -\det(A)$$

QED
• **Theorem** - If two rows (columns) of $A$ are equal, then $\det (A) = 0$.

• **Proof** - Suppose rows $r$ and $s$ of $A$ are equal. Interchange rows $r$ and $s$ to obtain matrix $B$. Then $\det (B) = -\det (A)$. However, $B = A$ so $\det (B) = \det (A)$. Thus $\det (A) = 0$.

QED
• **Theorem** - If a row (column) of $A$ consists entirely of zeros, then $\det(A) = 0$.

• **Proof** - Let the $i$th row of $A$ be all zeros. Each term of $\det(A)$ contains a factor from the $i$th row of $A$. So, $\det(A) = 0$.

QED
• **Theorem** - If $B$ is obtained from $A$ by multiplying a row (column) of $A$ by a real number $c$, then 
  \[ \det(B) = c \det(A) \].

• **Proof** - Suppose the $i$th row of $A$ is multiplied by $c$ to get $B$. Each term of $\det(B)$ contains a single factor from the $i$th row of $B$. Thus, each term of $\det(B)$ consists of a single factor of $c$ times the corresponding term of $\det(A)$. So, 
  \[ \det(B) = c \det(A) \].

QED
• **Theorem** - If $B = [b_{ij}]$ is obtained from $A = [a_{ij}]$ by adding to each element of $r$th row (column) of $A$, $c$ times the corresponding element of the $s$th row (column), $r \neq s$, of $A$, then $\det(B) = \det(A)$

• **Proof** - Note that $b_{ij} = a_{ij}$ for $i \neq r$. For row $r$ ($r \neq s$), have $b_{rj} = a_{rj} + c a_{sj}$. Without loss of generality, suppose that $r < s$. 
Proof (continued)

Then $\det(B) = \sum (\pm) b_{1j_1} b_{2j_2} \cdots b_{rj_r} \cdots b_{sj_s} \cdots b_{nj_n}$

$$= \sum (\pm) a_{1j_1} a_{2j_2} \cdots (a_{rj_r} + ca_{sj_r}) \cdots a_{sj_s} \cdots a_{nj_n}$$

$$= \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{rj_r} \cdots a_{sj_s} \cdots a_{nj_n}$$

$$+ \sum (\pm) a_{1j_1} a_{2j_2} \cdots ca_{sj_r} \cdots a_{sj_s} \cdots a_{nj_n}$$

$$= \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{rj_r} \cdots a_{sj_s} \cdots a_{nj_n}$$

$$+ c \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{sj_r} \cdots a_{sj_s} \cdots a_{nj_n}$$

First summation is $\det(A)$. Second summation is the determinant of a matrix having row $r$ equal to row $s$, so its determinant is 0. Thus $\det(B) = \det(A)$ QED
Comment

• The previous three theorems describe what happens to the determinant of a matrix $\mathbf{A}$ when the following elementary row (column) operations are performed on it
  
  – Type I - Interchange two rows (columns) of $\mathbf{A}$
  – Type II - Multiply row (column) $i$ of $\mathbf{A}$ by $c \neq 0$. (Theorem also covers the case $c = 0$)
  – Type III - Add $c$ times row (column) $i$ of $\mathbf{A}$ to row (column) $j$ of $\mathbf{A}$, $i \neq j$
• **Theorem** - Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be an upper (lower) triangular matrix, then $\det(A) = a_{11}a_{22} \ldots a_{nn}$. That is, the determinant of a triangular matrix is just the product of the elements on the main diagonal.

• **Proof** - Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be upper triangular, i.e. $a_{ij} = 0$ for $i > j$. Then $\det(A) = \sum (\pm) a_{1j_1}a_{2j_2} \cdots a_{nj_n}$

In each term, the factor $a_{ij_i} = 0$ if $i > j_i$. So the only surviving terms must have $i \leq j_i$ for all $i$. Since $j_1j_2 \ldots j_n$ is a permutation of $\{1, 2, \ldots, n\}$, there is a single surviving term $j_1 = 1, j_2 = 2, \ldots, j_n = n$. Therefore $\det(A) = a_{11}a_{22} \ldots a_{nn}$. QED
Comment

- The three elementary row operations are
  - Type I - Interchange two rows (columns) of $A$
  - Type II - Multiply row (column) $i$ of $A$ by $c \neq 0$. (Theorem also covers the case $c = 0$)
  - Type III - Add $c$ times row (column) $i$ of $A$ to row (column) $j$ of $A$, $i \neq j$

- Each elementary row operation may be accomplished by multiplying $A$ by the appropriate elementary matrix, which is obtained by performing the operation on the identity matrix $I$. By the preceding theorem, $\det(I) = 1$. Let $E_I$, $E_{II}$ and $E_{III}$ be elementary matrices of Type I, Type II and Type III respectively. Then, by the theorems proved recently

\[
\det(E_I) = -1 \quad \det(E_{II}) = c \quad \det(E_{III}) = 1
\]
• **Theorem** - Let $E$ be an elementary matrix. Then
\[
\det(EA) = \det(E) \det(A) \quad \text{and} \quad \det(AE) = \det(A) \det(E)
\]

• **Proof** -
  
  If $E$ is Type I, $\det(EA) = -\det(A)$. Since $\det(E) = -1$, $\det(EA) = \det(E) \det(A)$

  If $E$ is Type II, $\det(EA) = c \det(A)$. Since $\det(E) = c$, $\det(EA) = \det(E) \det(A)$

  If $E$ is Type III, $\det(EA) = \det(A)$. Since $\det(E) = 1$, $\det(EA) = \det(E) \det(A)$

  The proof for $\det(AE)$ follows from consideration of elementary column operations

**QED**
• **Theorem** - Let $A$ be a $n \times n$ matrix. Then $A$ is nonsingular if and only if $\det(A) \neq 0$

• **Proof** - $\implies$ Let $A$ be nonsingular. Then $A$ can be expressed as the product of elementary matrices, $A = E_1E_2 \cdots E_k$. Then $\det(A) = \det(E_1E_2 \cdots E_k) = \det(E_1) \det(E_2) \cdots \det(E_k) \neq 0$

$\impliedby$ Let $\det(A) \neq 0$. Suppose $A$ is singular. Then $A$ is row equivalent to a matrix $B$ with a row of zeros. $A = E_1E_2 \cdots E_k B$ where $E_1, E_2, \ldots, E_k$ are elementary matrices. Since $B$ has a row of zeros, $\det(B) = 0$. Now, $\det(A) = \det(E_1E_2 \cdots E_k B) = \det(E_1) \det(E_2) \cdots \det(E_k) \det(B) = 0$, which is a contradiction. So, $A$ is nonsingular. 

QED
Theorem - If $A$ is an $n \times n$ matrix, then $Ax = 0$ has a nontrivial solution if and only if $\det(A) = 0$.

Proof - $Ax = 0$ has a nontrivial solution if and only if $A$ is singular. If $A$ is singular, then $\det(A) = 0$.

QED
• **Theorem** - If \( A \) and \( B \) are \( n \times n \) matrices, then
\[
\det(AB) = \det(A) \det(B)
\]

• **Proof** - If \( A \) or \( B \) is singular, then \( AB \) is singular and the result follows immediately. So, suppose that \( A \) and \( B \) are both nonsingular. Then \( A \) and \( B \) can be expressed as the product of elementary matrices.

\[
A = E_1E_2 \cdots E_p \quad \text{and} \quad B = F_1F_2 \cdots F_q
\]

\[
\det(A) = \det(E_1) \det(E_2) \cdots \det(E_p)
\]

\[
\det(B) = \det(F_1) \det(F_2) \cdots \det(F_q)
\]

\[
\det(AB) = \det(E_1E_2 \cdots E_pF_1F_2 \cdots F_q) = \det(E_1) \det(E_2) \cdots \det(E_p) \det(F_1) \det(F_2) \cdots \det(F_q) = \det(A) \det(B)
\]

QED
• **Theorem** - If $A$ is nonsingular, then
  \[ \det(A^{-1}) = 1 / \det(A) \]

• **Proof** - Let $I_n$ be the $n \times n$ identity. Then $AA^{-1} = I_n$ and $\det(I_n) = 1$. By the preceding theorem, $\det(A) \det(A^{-1}) = 1$, so $\det(A^{-1}) = 1 / \det(A)$

\[ \text{QED} \]
• **Theorem** - If $A$ and $B$ are similar matrices, then $\det(A) = \det(B)$

• **Proof** - $A$ and $B$ are similar if there exists a nonsingular matrix $P$ such that $B = P^{-1}AP$. Then

$$
\det(B) = \det(P^{-1}AP) = \det(P^{-1}) \det(A) \det(P) = \det(A)
$$

QED
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• Comment - To this point, there has been only one way to calculate the determinant, i.e. from the definition

\[
\text{det}(A) = \sum (\pm) a_{j_1} a_{j_2} \cdots a_{j_n}
\]

There are other ways of calculating the determinant that can be useful for calculations and for proofs.
Defn - Let $A = [ \ a_{ij} \ ]$ be an $n \times n$ matrix. Let $M_{pq}$ be the $(n - 1) \times (n - 1)$ submatrix of $A$ obtained by deleting the $p$th row and $q$th column of $A$. The determinant $\det(M_{pq})$ is called the minor of $a_{pq}$.

Defn - Let $A = [ \ a_{ij} \ ]$ be an $n \times n$ matrix. The cofactor $A_{pq}$ of $a_{pq}$ is defined as $A_{pq} = (-1)^{p+q} \det(M_{pq})$. 
Determinants

Example

• Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$M_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \quad A_{11} = (-1)^2 \det(M_{11}) = -3$$

$$M_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad A_{12} = (-1)^3 \det(M_{12}) = 6$$

$$M_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}, \quad A_{31} = (-1)^4 \det(M_{31}) = -3$$
Comment

• Examine pattern of signs of term \((-1)^{p+q}\)

\[
\begin{bmatrix}
  + & - & + \\
  - & + & - \\
  + & - & + \\
\end{bmatrix} \quad \begin{bmatrix}
  + & - & + & - \\
  - & + & - & + \\
  + & - & + & - \\
  - & + & - & + \\
\end{bmatrix}
\]

\[n = 3 \quad n = 4\]

When using cofactors, don’t have to evaluate \((-1)^{p+q}\)

Just remember the pattern
• **Theorem** - Let $A = [ a_{ij} ]$ be an $n \times n$ matrix. Then

$$ \det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \quad \text{(expansion of } \det(A) \text{ with respect to row } i)$$

$$ \det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \quad \text{(expansion of } \det(A) \text{ with respect to column } j)$$
Linear Algebra

Determinants

Cofactor Expansion

Example

• Expand the determinant of a 3 x 3 matrix

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

\[
\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\]

Reorganize expression with respect to first row

\[
\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) \\
+ a_{13}(a_{21}a_{32} - a_{22}a_{31})
\]
Linear Algebra

Determinants

Cofactor Expansion

Example (continued)

Try to recognize the terms in parentheses

\[ A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}a_{33} - a_{23}a_{32}) \]

\[ A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (a_{21}a_{33} - a_{23}a_{31}) \]

\[ A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (a_{21}a_{32} - a_{22}a_{31}) \]

So \( \det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \)
Example (continued)

Regroup $\det(A)$ with respect to the first column of $A$

$$\det(A) = a_{11} (a_{22}a_{33} - a_{23}a_{32}) + a_{21} (a_{13} a_{32} - a_{12}a_{33}) + a_{31} (a_{12}a_{23} - a_{13}a_{22})$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}a_{33} - a_{23}a_{32})$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = (a_{13}a_{32} - a_{12}a_{33})$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = (a_{12}a_{23} - a_{13}a_{22})$$

So $\det(A) = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$
Example

- Evaluate $\det(A) = \begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}$

Pick row or column with large number of zeros, such as column 2
Example (continued)

\[ \det(A) = 2A_{12} + 2A_{22} + 0A_{32} + 0A_{42} \]

\[ = 2(-1)^{1+2} \begin{vmatrix} -4 & 1 & 3 \\ 3 & 0 & -3 \\ 2 & -2 & 3 \end{vmatrix} + 2(-1)^{2+2} \begin{vmatrix} 1 & -3 & 4 \\ 3 & 0 & -3 \\ 2 & -2 & 3 \end{vmatrix} \]

\[ = -2 \left( 1(-1)^{1+2} \left( 3 \cdot 3 - 2(-3) \right) + (-2)(-1)^{3+2} \left( (-4)(-3) - 3 \cdot 3 \right) \right) \]

\[ + 2 \left( (-3)(-1)^{1+2} \left( 3 \cdot 3 - 2(-3) \right) + (-2)(-1)^{3+2} \left( 1(-3) - 3 \cdot 4 \right) \right) \]

\[ = -2 \left( -(9 + 6) + 2(12 - 9) \right) + 2 \left( 3(9 + 6) + 2(-3 - 12) \right) \]

\[ = -2(-15 + 6) + 2(45 - 30) = -2(-9) + 2(15) = 48 \]
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• Have seen an explicit formula for inverse of a 2\times2 matrix

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix} = \begin{bmatrix}
d & -b \\
ad-bc & ad-bc \\
-c & a \\
ad-bc & ad-bc
\end{bmatrix}
\]

In terms of more general notation this is

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\]
• **Theorem** - Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = 0 \text{ for } i \neq k$$

$$a_{1j}A_{1k} + a_{2j}A_{2k} + \cdots + a_{nj}A_{nk} = 0 \text{ for } j \neq k$$

i.e. sum of row or column times cofactors of another row or column is 0

• **Proof** - Consider matrix $B$ obtained from $A$ by replacing $k$th row of $A$ with $i$th row of $A$. Matrix $B$ has two identical rows, the $i$th and the $k$th, so $\det(B) = 0$. Expand $\det(B)$ with respect to $k$th row. The elements of the $k$th row of $B$ are $a_{i1}, a_{i2}, \ldots, a_{in}$. So $0 = \det(B) = a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn}$. The second result follows from $\det(B^T) = \det(B)$ QED
• **Defn** - Let $A = [a_{ij}]$ be an $n \times n$ matrix. The matrix

$$
\begin{bmatrix}
A_{11} & A_{21} & \cdots & A_{n1} \\
A_{12} & A_{22} & \cdots & A_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
A_{1n} & A_{2n} & \cdots & A_{nn}
\end{bmatrix}
$$

is called the **transpose of the matrix of cofactors** or (book’s term) **adjoint** of $A$, notation $\text{adj}(A)$.
Example

Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} \)

\[
A_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 0 \end{vmatrix} = -48 \quad A_{12} = -\begin{vmatrix} 4 & 6 \\ 7 & 0 \end{vmatrix} = 42 \quad A_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3
\]

\[
A_{21} = -\begin{vmatrix} 2 & 3 \\ 8 & 0 \end{vmatrix} = 24 \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 0 \end{vmatrix} = -21 \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 6
\]

\[
A_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3 \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6 \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3
\]
Example (continued)

\[
\text{adj}(A) = \begin{bmatrix}
A_{11} & A_{21} & A_{31} \\
A_{12} & A_{22} & A_{32} \\
A_{13} & A_{23} & A_{33}
\end{bmatrix} = \begin{bmatrix}
-48 & 24 & -3 \\
42 & -21 & 6 \\
-3 & 6 & -3
\end{bmatrix}
\]

\[
\text{det}(A) = 7A_{31} + 8A_{32} = 7(-3) + 8(6) = 27
\]

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 0
\end{bmatrix}
\begin{bmatrix}
-48 & 24 & -3 \\
42 & -21 & 6 \\
-3 & 6 & -3
\end{bmatrix} =
\begin{bmatrix}
27 & 0 & 0 \\
0 & 27 & 0 \\
0 & 0 & 27
\end{bmatrix}
\]
Linear Algebra

Determinants

• **Theorem** - Let \( A = [ a_{ij} ] \) be an \( n \times n \) matrix. Then
\[
A \left( \text{adj}(A) \right) = \left( \text{adj}(A) \right) A = \det(A) \mathbf{I}_n
\]

• **Proof** - Consider the elements of the matrix that is the product of \( A \) and \( \text{adj}(A) \).

\[
A \left( \text{adj}(A) \right) = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
    A_{11} & A_{21} & \cdots & A_{n1} \\
    A_{12} & A_{22} & \cdots & A_{n2} \\
    \vdots & \vdots & \ddots & \vdots \\
    A_{1n} & A_{2n} & \cdots & A_{nn}
\end{bmatrix}
\]

The \((i, j)\) entry of \( A \left( \text{adj}(A) \right) \) is just the \( i \) th row of \( A \) times the \( j \) th column of \( \text{adj}(A) \).
Proof (continued)

From two earlier theorems

\[
a_{i_1} A_{j_1} + a_{i_2} A_{j_2} + \cdots + a_{i_n} A_{j_n} = \begin{cases} 
\det(A) & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]

\[
A(\text{adj}(A)) = \begin{bmatrix}
\det(A) & 0 & \cdots & 0 \\
0 & \det(A) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \det(A)
\end{bmatrix} = \det(A)I_n
\]
• **Proof** (continued)

By a similar argument, the \((i, j)\) entry of \((\text{adj}(A))A\) is

\[
A_{1i}a_{1j} + A_{2i}a_{2j} + \cdots + A_{ni}a_{nj} = \begin{cases} 
\det(A) & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]

so \((\text{adj}(A))A = \det(A)I_n\)

QED
**Theorem** - Let $A$ be an $n \times n$ matrix with $\det(A) \neq 0$. Then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{bmatrix}
\frac{A_{11}}{\det(A)} & \frac{A_{21}}{\det(A)} & \cdots & \frac{A_{n1}}{\det(A)} \\
\frac{A_{12}}{\det(A)} & \frac{A_{22}}{\det(A)} & \cdots & \frac{A_{n2}}{\det(A)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{A_{1n}}{\det(A)} & \frac{A_{2n}}{\det(A)} & \cdots & \frac{A_{nn}}{\det(A)}
\end{bmatrix}$$

**Proof** - By the preceding theorem, $A \left( \text{adj}(A) \right) = \det(A)I_n$ then $\text{adj}(A) = \det(A)A^{-1}$

and $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

QED
Example

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 0 \\
\end{bmatrix}^{-1} = \frac{1}{27} \begin{bmatrix}
-48 & 24 & -3 \\
42 & -21 & 6 \\
-3 & 6 & -3 \\
\end{bmatrix} = \frac{1}{9} \begin{bmatrix}
-16 & 8 & -1 \\
14 & -7 & 2 \\
-1 & 2 & -1 \\
\end{bmatrix}
\]
Linear Algebra

Determinants

Topics

• Preliminaries
• Definition
• Properties of Determinants
• Cofactor Expansion
• Inverse of a Matrix
• Other Applications of Determinants
• Computing Determinants
• Theorem - (Cramer’s Rule) Consider the linear system of $n$ equations in $n$ unknowns

\[a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1\]
\[a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2\]
\[\vdots \quad \vdots \quad \vdots \quad \vdots \]
\[a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n\]

Let $A = [a_{ij}]$ be the coefficient matrix, so that the system may be expressed as $Ax = b$. If $\det(A) \neq 0$, then the system has a unique solution

\[x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \cdots \quad x_n = \frac{\det(A_n)}{\det(A)}\]

Where $A_i$ is the matrix obtained from $A$ by replacing its $i$th column with $b$
• **Proof** - If \( \det(A) \neq 0 \), then \( A \) is nonsingular. So

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}b = \begin{bmatrix} \frac{A_{11}}{\det(A)} & \frac{A_{21}}{\det(A)} & \ldots & \frac{A_{n1}}{\det(A)} \\ \frac{A_{12}}{\det(A)} & \frac{A_{22}}{\det(A)} & \ldots & \frac{A_{n2}}{\det(A)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{A_{1n}}{\det(A)} & \frac{A_{2n}}{\det(A)} & \ldots & \frac{A_{nn}}{\det(A)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
\]

\[
x_i = \frac{A_{1i}}{\det(A)} b_1 + \frac{A_{2i}}{\det(A)} b_2 + \ldots + \frac{A_{ni}}{\det(A)} b_n \quad 1 \leq i \leq n
\]
• **Proof** (continued)

Define \( A_i \) as \( A \) with the \( i \)th column replaced by \( b \)

\[
A_i = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1(i-1)} & b_1 & a_{1(i+1)} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2(i-1)} & b_2 & a_{2(i+1)} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{n(i-1)} & b_n & a_{n(i+1)} & \cdots & a_{nn}
\end{bmatrix}
\]

Evaluate \( \det(A_i) \) by expanding with respect to \( i \)th column, then

\[
\det(A_i) = A_{1i}b_1 + A_{2i}b_2 + \cdots + A_{ni}b_n
\]

So

\[
x_i = \frac{\det(A_i)}{\det(A)} \quad 1 \leq i \leq n
\]

QED
Linear Algebra

Determinants

Example

- Solve the system

\[
\begin{align*}
x_1 + 2x_2 + 3x_3 &= 6 \\
2x_1 - 3x_2 + 2x_3 &= 14 \\
3x_1 + x_2 - x_3 &= -2
\end{align*}
\]

Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix} \) be the coefficient matrix.

The determinant of \( A \) is:

\[
\det(A) = 50
\]

Thus, the solutions are:

\[
x_1 = \frac{\det(A)}{\det(A)} = \frac{50}{50} = 1
\]

\[
x_2 = \frac{\det(A)}{\det(A)} = \frac{-100}{50} = -2
\]

\[
x_3 = \frac{\det(A)}{\det(A)} = \frac{150}{50} = 3
\]
Linear Algebra

Determinants

Topics

• Preliminaries
• Definition
• Properties of Determinants
• Cofactor Expansion
• Inverse of a Matrix
• Other Applications of Determinants
• *Computing Determinants*
Linear Algebra

Computing Determinants

• It is practical to use determinants for computations involving an $n \times n$ matrix $A$ when $n \leq 4$

• For larger systems, other methods such as Gaussian elimination run much more quickly