Math 266 Fourth Exam Review Notes

Alternating Series Test

If (i) $a_k \geq a_{k+1} > 0$ and (ii) $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is convergent.

Error in using S_n to approximate the sum S of a convergent alternating series is $|S - S_n| \leq a_{n+1}$

A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent. (ex. $\sum (-1)^n 1/n^2$)
A series $\sum a_n$ is **conditionally convergent** if $\sum a_n$ is convergent and $\sum |a_n|$ is divergent. (ex. $\sum (-1)^n 1/n$)

$\sum a_n$ absolutely convergent $\Rightarrow$ $\sum a_n$ convergent

Ratio Test (Root Test) for Absolute Convergence

Let $\sum a_n$ be a series of nonzero terms, and let $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ $\left(\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \right)$.

(i) $L < 1 \Rightarrow \sum a_n$ is absolutely convergent.

(ii) $L > 1$ or $\infty \Rightarrow \sum a_n$ is divergent.

(iii) $L = 1$, use a different test.

A **power series in *x*** is a series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$, a_k real

A **power series in *x – c*** is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \ldots + a_n (x-c)^n + \ldots, \ a_k \text{ real}$$

If $\sum a_n x^n$ is a power series in *x*, then exactly one of the following is true:

(i) The series converges only if $x = 0$

(ii) The series is absolutely convergent for every *x.*

(iii) There is a number $r > 0$ such that the series is absolutely convergent if $x \in (-r, r)$ and divergent if $x < -r$ or $x > r$; *r* is called the **radius of convergence** of the series.

The **interval of convergence** is the totality of numbers for which a power series converges.

If $\sum a_n (x-c)^n$ is a power series in *x* - *c*, then exactly one of the following is true:

(i) The series converges only if $x = c$

(ii) The series is absolutely convergent for every *x.*

(iii) There is a number $r > 0$ such that the series is absolutely convergent if $x \in (c-r, c+r)$ and divergent if $x < c - r$ or $x > c + r$; *r* is called the **radius of convergence** of the series.

Power Series Representations of Functions

Suppose $\sum a_n x^n$ has a radius of convergence $r > 0$, and let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

for every $x \in (-r, r)$. If $x \in (-r, r)$, then

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$\int_0^x f(t)dt = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \ldots + a_n \frac{x^{n+1}}{n+1} + \ldots = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Maclaurin Series for *f(x)*

If a function *f* has a power series representation $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $r > 0$,

then $f^k(0)$ exists for every positive integer *k* and $a_n = \frac{f^{(n)}(0)}{n!}$. Thus,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

Taylor Series for *f(x)*

If a function *f* has a power series representation $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ with radius of convergence $r > 0$,

then $f^k(c)$ exists for every positive integer *k* and $a_n = \frac{f^{(n)}(c)}{n!}$. Thus,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

nth-degree Taylor polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

nth-degree Maclaurin polynomial (replace *c* by 0)

Taylor's Formula with Remainder

Let *f* have n + 1 derivatives throughout an interval containing *c*. If *x* is any number in the interval, $x \neq c$, then there is a number *z* between *c* and *x* such that

$$f(x) = P_n(x) + R_n(x), \quad \text{where } R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

If $\lim_{n\to\infty} R_n(x) = 0$, then $f(x)$ is represented by the Taylor series for $f(x)$ at *c*.

Important Limit: $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$

Important Maclaurin Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \text{ for } x \in (-\infty, \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \text{ for } x \in (-\infty, \infty)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \text{ for } x \in (-\infty, \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^n \frac{x^{n+1}}{n+1} + \ldots \text{ for } x \in (-1, 1]$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots \text{ for } x \in [-1, 1]$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots \text{ for } x \in (-\infty, \infty)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots \text{ for } x \in (-\infty, \infty)$$

Binomial Series If $|x| < 1$, then for every real number *k*,

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \ldots + \frac{k(k-1)\cdots(k-n+1)}{n!}x^n + \ldots$$